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Valuations of Lattice-Ordered Groups

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In this paper, we introduce the concept of a valuation mapping of an l-group G onto a distributive lattice and use such valuations to investigate the structure of G. Then we examine the maximal immediate extensions of G with respect to these valuations. For the natural valuation, these are the archimedean extensions (a-extensions) first investigated by P. Conrad (1966, *J. Indiana Math. Soc.* **30**, 131–160) and S. Wolfenstein (1970, dissertation, University of Paris). This leads to new results and new proofs of old results about a-extensions. Then we obtain new structure theorems for Δ-extensions of l-groups. In another paper, it will be shown that this valuation theory determines *all* the torsion classes of l-groups that have invariant torsion radicals. This includes most of the well known and interesting torsion classes that are not l-varieties. © 1997 Academic Press

1. INTRODUCTION

The study of valuations on algebraic objects (fields, rings, and groups) has its origins in work done by Krull [25] and Kaplansky [21, 22]. Conrad [11] generalized the definition from a map of a field into an ordered

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abelian group to a function from an abelian operator group into a partially ordered set, and used this generalization to provide a shorter proof of the Hahn Embedding Theorem. Gravett [18] gave a second generalization to the case of a linear space over a field K into a totally set Δ, and showed that this generalization implied the work of Conrad [11]. We refer the interested reader to the cited papers by Gravett [18], Kaplansky [21, 22], and Krull [25] and also to the excellent presentation of field valuations by Schilling [29] and in the third volume of Jacobson [20].

In this paper, we in a sense return to the earlier work of Conrad and study valuations on lattice-ordered groups, where the codomain of the valuation map can be taken to be a lattice of order ideals of the root system of regular subgroups. By studying extensions based on such valuations, we obtain new and interesting results on archimedean extensions and archimedean closures of arbitrary lattice-ordered groups.

We review now some of the basic terms and concepts of lattices and lattice-ordered groups. A *lattice* is a partially order set L such that for every pair of elements $x, y \in L$, there exists a least upper bound (called the *join* and written $x \vee y$) and a greatest lower bound (called the *meet* and written $x \wedge y$). An *ideal* of a partially ordered P is a subset S such that if $x \leq s \in S$, then $x \in S$, while a *dual ideal* (*filter*) is a subset T such that if $y \geq t \in T$, then $y \in T$. An ideal I of a lattice is *prime* if $x \wedge y \in I$ implies either $x \in I$ or $y \in I$. Within a given partially ordered set P, an element a *covers* an element b if $b < a$ and there are no intervening elements.

A *lattice-ordered group* (written l-group), is a group G whose underlying set is a lattice such that if $g \leq h$, then for any $x, y \in G$, $xgy \leq xhy$. For an l-group G, G^+ denotes the set $\{g \in G : g \geq e\}$. For $g \in G$, the *positive part* of g (written g_+) is $g \vee e$, while the *negative part* of g (written g_-) is $g^{-1} \vee e$; the *absolute value* of g (written $|g|$) is $g_+ \vee g_- = g_+ g_- = g \vee g^{-1}$. Two elements $a, b \in G$ are *disjoint* if $|a| \wedge |b| = e$. An l-group G is *archimedean* if for any pair of positive elements $a, b \in G^+$, there exists an integer n such that $a^n \nless b$; an archimedean l-group is necessarily abelian.

An *l-subgroup* A of an l-group G is both a sublattice and a subgroup. An l-subgroup C is *convex* if $e \leq x \leq c \in C$ implies $x \in C$; $\mathcal{C}(G)$ will denote the lattice of convex l-subgroups of G, partially ordered by inclusion. For $C \in \mathcal{C}(G)$, the (right) cosets of C are ordered by $Cx \leq Cy$ if there exists $c \in C$ such that $cx \leq y$. A normal convex l-subgroup L is called an *l-ideal*; under the quotient group operation and coset order, G/L is an l-group. $\mathcal{C}(G)$ is a complete sublattice of the lattice of subgroups of G. Thus for any element $g \in G$, there exists a minimal convex l-subgroup containing g; this is called a *principal* convex l-subgroup and is denoted $G(g)$. $\mathcal{C}_p(G)$

will denote the lattice of principal convex l-subgroups of G, partially ordered by inclusion.

A convex l-subgroup P of an l-group G is *prime* if P^+ is a prime lattice ideal of G^+; a convex l-subgroup is prime if and only if its right cosets are totally ordered under the coset order. Under inclusion, the prime convex l-subgroups of an l-group form a *root system*: i.e., a partially ordered set in which no two incomparable elements have a lower bound. A convex l-subgroup M maximal with respect to not containing an element g is *regular* and is a *value* of g. Regular convex l-subgroups are necessarily prime subgroups; $\Gamma(G)$ will denote the root system of regular convex l-subgroups of G, partially ordered by inclusion. The intersection of all convex l-subgroups properly containing a regular convex l-subgroup M also properly contains M, and is called the *cover* of M. It is customary to denote a regular convex l-subgroup by G_γ and its cover by G^γ. If $G_\gamma \lhd G^\gamma$, G_γ is called a *normal* value. G is *normal-valued* if every regular subgroup is a normal value. An element $s \in G$ is *special* if s has exactly one element; in this case, its value is called *special* as well. A special value is necessarily a normal value. An element $g \in G$ is *finite-valued* if g has one finitely many values; in this case, each value of g is a special value. G is *finite-valued* if every element of G is finite-valued. A *plenary subset* of $\Gamma(G)$ is a dual ideal Δ such that $\bigcap \Delta = (e)$. Given a root system Δ and an o-group I, $V(\Delta, I)$ will denote those elements of $\prod_\Delta I$ whose support satisfies the ascending chain conditions, and where $v \in V(\Delta, I)$ is positive if $v(\delta) > e$ for each maximal element $\delta \in \operatorname{supp}(v)$. With this ordering and with pointwise group operations, $V(\Delta, I)$ is an l-group.

For $x \subseteq G$, the *polar* of X, written $X^\perp$, $= \{g \in G : |g| \wedge |x| = e$ for all $x \in X\}$. $X^\perp \in \mathbf{C}(G)$. If $C \in \mathbf{C}(G)$ and G is the (group) direct sum of C and $C^\perp$, we write $G = C \boxplus C^\perp$. In this case, G is called the *cardinal sum* of C and $C^\perp$ and C is a *cardinal summand* of G. More generally, for a set $\{A_\lambda\}_\Lambda$ of l-groups, the *cardinal product*, denoted $\prod_\Lambda A_\lambda$ is the (external) group direct product of $\{A_\lambda\}_\Lambda$ with pointwise order operations. The *cardinal sum*, written $\sum_\Lambda A_\lambda$, is the l-subgroup of $\prod_\Lambda A_\lambda$ of those elements having finite support.

Two elements $x, y \in G$ are *a-equivalent* if there exist positive integers m and n such that $x < y^n$ and $y < x^m$. G is an *a-extension* of an l-subgroup A if for each $g \in G$, there exists $a \in A$ such that a is a-equivalent to g. If G is an a-extension of A, then $\mathbf{C}(G) \cong \mathbf{C}(A)$ by way of the map $C \to C \cap A$. If an l-group G has no proper a-extensions, G is called *a-closed*. Every l-group G has an a-extension H which is a-closed; H is called an *a-closure* of G.

A *torsion class* $\mathbf{T}$ of l-groups is a class that is closed with respect to containing convex l-subgroups, closed with respect to l-homomorphic images, and also has the following property: for any l group G, if $\{C_\lambda\}$ is a set of convex l-subgroups of G that are also in T, then $\bigvee C_\lambda \in \mathsf{T}$. An l-variety is an equationally defined class of l-groups; an l-variety is necessarily a torsion class.

Throughout, $\mathbb{Z}$ will denote the integers, $\mathbb{Q}$ the rationals, and $\mathbb{R}$ the reals, all with the usual addition and order.

2. VALUATIONS OF LATTICE-ORDERED GROUPS

DEFINITION 1. Let Δ be a root system; T_Δ will denote the set of all trivially ordered subsets (antichains) of Δ. For $X, Y \in T_\Delta$, define $X \leq Y$ if for any $x \in X$, there exists $y \in Y$ such that $x \leq y$ in Δ.

PROPOSITION 2.1. *T_Δ is a distributive lattice, where for $X, Y \in T_\Delta$, $X \vee Y = \{$maximal elements in $X \cup Y\}$ and $X \wedge Y = \{z \in X \cup Y \mid$ there exist $x \in X$ and $y \in Y$ such that $z \leq x$ and $z \leq y\}$.*

Proof. One can prove this directly. However, for each $X \in T_\Delta$, let c_X be the characteristic function on X in the l-group $V(\Delta, \mathbb{Z})$. Then clearly $\{c_X : X \in T_\Delta\}$ is a distributive sublattice of $V(\Delta, \mathbb{Z})$ and $X \to c_X$ is a lattice isomorphism of T_Δ onto $\{c_X : X \in T_\Delta\}$. ▮

DEFINITION 2. Let G be an l-group and let Δ be a plenary subset of the root system $\Gamma = \Gamma(G)$ of the regular subgroups of G. For $g \in G$, Δ_g will denote the set of all values of g in Δ. Since $\Delta_g = \Delta_{|g|}$, we need only consider $g \in G^+$. Δ_G will denote $\{\Delta_g : g \in G^+\}$.

PROPOSITION 2.2. *Δ_G is a sublattice of the lattice T_Δ. Moreover, for any $a,b \in G^+$, $\Delta_a \vee \Delta_b = \Delta_{(a \vee b)} = \Delta_{ab}$ and $\Delta_a \wedge \Delta_b = \Delta_{(a \wedge b)}$.*

Proof. $a \vee b \in G^\gamma \setminus G_\gamma$ if and only if $a, b \in G^\gamma$ and $a \notin G_\gamma$ or $b \notin G_\gamma$, which is if and only if γ is a value of a or b and also maximal in $\Delta_a \cup \Delta_b$, and this is if and only if $\gamma \in \Delta_a \vee \Delta_b$. Thus $\Delta_a \vee \Delta_b = \Delta_{(a \vee b)} = \Delta_{ab}$.

$\gamma \in \Delta_{(a \wedge b)}$ implies that $a \notin G_\gamma$ and $b \notin G_\gamma$. Thus a has a value $a \geq \gamma$ in Δ and b has a value $\beta \geq \gamma$ in Δ. Since Δ is a root system, $\alpha \geq \beta = \gamma$ or $\beta \geq \alpha = \gamma$. So $\gamma \in \Delta_a \wedge \Delta_b$, and hence $\Delta_{(a \wedge b)} \subseteq \Delta_a \wedge \Delta_b$. Conversely, if $\gamma \in \Delta_a \wedge \Delta_b$, then $\gamma = \alpha \leq \beta$ or $\gamma = \beta \leq \alpha$, where $\alpha \in \Delta_a$ and $\beta \in \Delta_b$. So $\gamma \in \Delta_{(a \wedge b)}$ and thus $\Delta_a \wedge \Delta_b \subseteq \Delta_{(a \wedge b)}$. ▮

The above proof clearly gives:

COROLLARY 2.3. *The map $\pi_\Delta : G^+ \to \Delta_G : g \to \Delta_g$ is a lattice homomorphism.*

DEFINITION 3. The lattice homomorphism π_Δ is the Δ-*valuation* of G. If $\Delta = \Gamma$, this valuation will be called the *natural valuation*: $g \to \Gamma_g$.

Note that $\Delta_g \to c_{\Delta_g}$ is a lattice isomorphism of Δ_G into $V(\Delta, \mathbb{Z})^+$. So $g \to c_{\Delta_g}$ is a lattice homomorphism of G^+ into $V(\Delta, \mathbb{Z})^+$.

COROLLARY 2.4. *The map* $\tau: \Delta_G \to \mathcal{C}_p(G): \Delta_g \to G(g)$ *is a lattice homomorphism. The map* $\Gamma_g \to G(g)$ *is a lattice isomorphism of* Γ_g *onto* $\mathcal{C}_p(G)$.

Proof. $(\Delta_a \vee \Delta_b) = (\Delta_{(a \vee b)}\tau = G(a \vee b) = G(a) \vee G(b) = \Delta_a\tau \vee \Delta_b\tau$, and $(\Delta_a \wedge \Delta_b)\tau = (\Delta_{(a \wedge b)})\tau = G(a \wedge b) = G(a) \cap G(b) = \Delta_a\tau \wedge \Delta_b\tau$.

Finally, Wolfenstein [30] proved that $\Gamma_a = \Gamma_b$ if and only if $G(a) = G(b)$, which proves the second assertion. ∎

The following definition is a straightforward generalization of similar definitions by Krull [25], Gravett [18], and Schilling [29].

DEFINITION 4. Let $G \subseteq H$ be l-groups and $\sigma: G \to \Delta$ be a valuation on G. A valuation $\tau: H \to \Lambda$ is an *immediate extension* of σ if (i) $\Lambda = \Delta$, (ii) for all $g \in G$, $\tau(g) = \sigma(g)$, and τ is the unique extension of σ to H.

Suppose that H is an l-subgroup of G; let $\Gamma_H = \{\Gamma_h : h \in H^+\}$. Γ_H is then easily seen to be a sublattice of Γ_G and $\Gamma_H = \Gamma_G$ if and only if G is an a-extension of H. Thus for the natural valuation, immediate extensions are a-extensions and maximal immediate extensions are a-closures.

THEOREM 2.5. *For an l-group G, the lattice* $\mathcal{C}(G)$ *is isomorphic to the lattice* $\widehat{\Gamma_G}$ *of ideals of the lattice* Γ_G.

Moreover, under this isomorphism, principal convex l-subgroups are mapped onto principal ideals of Γ_G; *prime subgroups are mapped onto prime ideals of* Γ_G; *and regular subgroups are mapped onto prime ideals of* Γ_G *which have a minimal ideal properly containing them.*

Proof. For $C \in \mathcal{C}(G)$, define $C\alpha$ to be $\Gamma_C = \{\Gamma_c : c \in C\}$. For $I \in \widehat{\Gamma_G}$, define $I\beta$ to be $\{g \in G : \Gamma_g \in I\}$. Now if $\Gamma_x \le \Gamma_x$ for $e < c \in C$ and $e < x \in G$, then $\Gamma_x = \Gamma_x \wedge \Gamma_c = \Gamma_{(x \wedge c)}$. Since $e \le x \wedge c \le c$, $x \wedge c \in C$. Now since x is a-equivalent to $x \wedge c$, $x \in C$. So $C\alpha$ is an ideal of Γ_G. Likewise, suppose that I is an ideal of Γ_G and that $\Gamma_a, \Gamma_b \in I$. Then $\Gamma_{ab} = \Gamma_a \vee \Gamma_b \in I$, and so the set of positive elements of $I\beta$ form a semigroup. If $e \le x \le g \in I\beta$, then $\Gamma_x = \Gamma_{(x \wedge g)} = \Gamma_x \wedge \Gamma_g \le \Gamma_f$ implies that $\Gamma_x \in I$ and so $x \in I\beta$. We thus have a convex subsemigroup D of positive elements, and so the convex l-subgroup generated by $D = \{g \in G : |g| \in D\} = I\beta$. Note that for $C \in \mathcal{C}(G)$, $C\alpha\beta = \{g \in G : \Gamma_g \in C\alpha\} = \{g \in G :$ there exists $c \in G$ such that $\Gamma_g = \Gamma_c\} = C$, and for $I \in \widehat{\Gamma_G}$, $I\beta\alpha = \{\Gamma_g : g \in {}_I\beta\} = \{\Gamma_g : \Gamma_g \in I\} = I$. So α and β are bijections.

Clearly if I, J are ideals of Γ_G such that $I \subseteq J$, then $I\beta \subseteq J\beta$. Likewise, if $C \subset D$ are convex l-subgroups of G, then $C\alpha \subseteq D\alpha$. So α and β preserve order and thus are lattice isomorphisms.

For $g \in G$, $G(g)\alpha$ is clearly mapped to the ideal $I = \{\Gamma_h : \Gamma_h \leq \Gamma_g\}$, and vice versa.

Now let P be a prime subgroup of G and suppose $e \leq a, b \in G$ such that $\Gamma_a \wedge \Gamma_b \in P\alpha$. Then $a \wedge b \in P\alpha\beta = P$, and since P is prime, either $a \in P$ or $b \in P$. But then either Γ_a or Γ_b is in $P\alpha$; so $P\alpha$ is a prime lattice ideal. The proof of the converse is virtually the same argument, while the proof of the correspondence between regular subgroups of G and prime ideals I having a minimal proper covering ideal is now clear. ∎

One set of convex l-subgroups that is easily distinguished in the above correspondence is the set of polars.

PROPOSITION 2.6. *Let Δ be a plenary subset of $\Gamma(G)$ and let $X \subseteq G^+$. Let $I_X = \{\Delta_t : \text{there exist } \Delta_x \in \Delta_X \text{ such that } \Delta_t \leq \Delta_x\}$ be the ideal of G_Δ generated by Δ_X. Then $X^\perp = \{g \in G : \Delta_g \text{ is incomparable to all } \Delta_t \in I_X\}$.*

Anderson, Conrad, and Martinez [3] gave a long list of properties of G that are determined by the lattice $C(G)$ and thus now by the lattice $\widehat{\Gamma_G}$. On the other hand, an example by Kenoyer [23] shows that $\widehat{\Gamma_G}$ cannot be used to determine if G is abelian, archimedean, or normal-valued, or if G belongs to any proper l-variety.

In another paper, the natural valuation will be used to produce *all* the torsion classes T of l-groups such that for any l-group G, the radical $T(G)$ is an invariant of $C(G)$.

We remark here that the natural valuation of an l-group G can be recovered from the lattice $C(G)$ by noting that $g \in G^\gamma \setminus G_\gamma$ if and only if $G(g) \subseteq G^\gamma$ and $G(g) \not\subset G_\gamma$. Thus $G(g)$ determines Γ_g, and thus $C_p(G) \cong \Gamma_G$ as lattices. We also now obtain that $C(G)$, in addition to being isomorphic to the lattice of ideals of Γ_G is isomorphic to the lattice of ideals of $C_p(G)$. We sum this up in the following theorem:

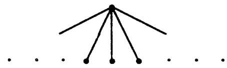
THEOREM 2.7. *The lattice $C(G)$ determines the root system Γ and the sublattice Γ_G of T_Γ.*

Thus the map $g \to G(g)$ is essentially the natural valuation of G, where $G(a) \subseteq G(b)$ if and only if $a < b^n$ for some positive integer n, which is if and only if $\Gamma_a \leq \Gamma_b$.

But $g \to \Gamma_g$ *is what a valuation should look like!* (in the sense that things are easier to "see" using this form of the natural valuation).

One should note here that for an arbitrary l-group G, to capture $C(G)$, it is generally necessary to use the ideals of Γ_G rather than the ideals of $\Gamma(G)$ itself. Indeed, as shown by Conrad [13], the ideals of $\Gamma(G)$ uniquely determine convex l-subgroups of G if and only if G is finite-valued.

However, that G is finite-valued or has Property (F) (no bounded subset is infinite) can naturally and rather obviously be characterized in terms of the natural valuation:



PROPOSITION 2.8. (a) *An l-group G is finite-valued if and only if each Γ_g is finite.*

(b) *An l-group G satisfies Property* (F) *if and only if each Γ_g is finite and Γ_G contains no copy of*

PROPOSITION 2.9. *For an l-group G, $\Gamma_G = T_\Gamma$ if and only if Γ has only finitely many roots.*

Proof. For any $\gamma \in \Gamma$, $\{\gamma\} \in T_\Gamma$ implies that if $\Gamma_G = T_\Gamma$, there exists $g \in G$ such that $\Gamma_g = \gamma$, which is to say that G_γ is a special value. Since every value is special, G must be finite-valued. But since $\Gamma_G = T_\Gamma$, T_Γ consists of only finite sets, and thus Γ has only a finite number of roots.

The reverse direction is clear. ∎

We can also generalize Theorem 2.5 from the natural valuation to an arbitrary valuation:

DEFINITION 5. Let G be an l-group and let Δ be a plenary subset of $\Gamma(G)$. $C \in \mathcal{C}(G)$ is Δ-*convex* if for any $c \in C$, $\Delta_x \leq \Delta_c$ implies $x \in C$.

THEOREM 2.10. *The set $\mathcal{C}_\Delta(G)$ of all the Δ-convex convex l-subgroups of G is a sublattice of $\mathcal{C}(G)$ and there exists a natural isomorphism α of $\mathcal{C}_\Delta(G)$ onto the lattice $\widehat{\Delta_G}$ of all the ideals of Δ_G.*

3. Γ-EXTENSIONS AND Γ-CLOSURES

PROPOSITION 3.1 (Conrad [12]; Wolfenstein [30]). *For positive elements x, y of an l-group G, the following are equivalent*:

(a) *There exist positive integers m and n such that $x \leq y^m$ and $y \leq x^n$.*

(b) $G(x) = G(y)$.

(c) $\Gamma_x = \Gamma_y$.

If any (and so all) of these conditions are met, then we say that x and y are a-equivalent or Γ-equivalent.

PROPOSITION 3.2 (Conrad [12]; Wolfenstein [30]). *For an l-subgroup G of an l-group H, the following are equivalent*:

(a) *The map $C \to C \cap G$ is a bijection of $\mathbf{C}(H)$ onto $\mathbf{C}(G)$.*

(b) *The map $C \to H(C)$ is a bijection of $\mathbf{C}(G)$ onto $\mathbf{C}(H)$, where $H(C)$ is the convex l-subgroup of H generated by C.*

(c) *For each $h \in H$, there exists $g \in G$ such that $\Gamma_g = \Gamma_h$.*

DEFINITION 6. If an l-subgroup G of an l-group H satisfies any (and so all) of the conditions of Proposition 3.2, then H is an *a-extension* (*Γ-extension*) of G, and G is an a-subgroup of H. If in addition, H admits no proper a-extensions, then H is an *a-closure* (*Γ-closure*) of G.

Note that if H is an a-extension of G, then $H_\gamma \to H_\gamma \cap G$ and $H^\gamma) \to H^\gamma \cap G$ are, respectively, isomorphisms of the root system $\Gamma(H)$ onto $\Gamma(G)$ and covers of regular subgroups in $\Gamma(H)$ to covers of regular subgroups in $\Gamma(G)$. Moreover, as shown by Byrd [8] and Wolfenstein [30], $H_\gamma \triangleleft H^\gamma$ if and only if $H_\gamma \cap G \triangleleft H^\gamma \cap G$. Thus for an a-subgroup G of an l-group H, G is normal-valued if and only if H is normal-valued.

Khuon [24] proves that a-closures exist for all l-groups, while W. C. Holland [19] was the first to give an example of an l-group with nonisomorphic a-closures. Conrad [12] proves that every abelian l-group has a maximal abelian a-extension. Wolfenstein [30] then proved that if an abelian l-group G has not proper abelian a-extensions, G is then a-closed. Thus every abelian l-group has an a-closure which is abelian. However, there do exist abelian l-groups having nonisomorphic abelian a-closures.

The lack of uniqueness carries over even into the case of archimedean l-groups. Any a-extension of an archimedean l-group must be archimedean, but Conrad [12] showed that $\prod_{i=1}^{\infty} \mathbb{Z}_i$ has non-isomorphic a-closures. Note that $\prod_{i=1}^{\infty} \mathbb{R}$ is *not* an a-extension of $\prod_{i=1}^{\infty} \mathbb{Z}$; however, $\prod_{i=1}^{\infty} \mathbb{R}$ is an a-extension of $\prod_{i=1}^{\infty} \mathbb{Q}$.

On the other hand, Conrad [12] showed that for any topological space X, $C(X)$ (the l-group of continuous real-valued functions on X with pointwise operations) is a-closed, and that, if X is an extremally disconnected Stone space, then $D(X) = \{f: X \to [-\infty, \infty] \mid f^{-1}(\mathbb{R})$ is a dense open set$\}$ (with pointwise operations on dense open sets) is also a-closed.

In this section, we wish to examine a-closed l-groups and those l-groups that admit unique a-closures. As indicated above, most of the research in this area has been directed toward abelian a-closures of abelian l-groups, and toward identifying abelian l-groups that have unique a-closures. The only result known to the authors about non-abelian l-groups with unique a-closures is the following theorem due separately to Khuon [24] and

Wolfenstein [30]:

THEOREM 3.3. *Let T be a chain such that* $\operatorname{Aut}(T)$, *the l-group of order-preserving permutations of T, is doubly transitive. Then the unique a-closure of* $\operatorname{Aut}(T)$ *is* $\operatorname{Aut}(\overline{T})$, *where $\overline{T}$ is the Dedekind completion of T.*

Now let H be an a-extension of an l-group G; for $C \in \mathbf{C}(G)$, let $H(C)$ denote the convex l-subgroup of H generated by C. Since the mapping $\mu\colon \mathbf{C}(G) \to \mathbf{C}(H) : C \to H(C)$ is a lattice isomorphism, obviously if C is recognizable in $\mathbf{C}(G)$, then $H(C)$ will share the same recognition. Thus μ maps principal convex l-subgroups (which are precisely the compact elements) onto principal convex l-subgroups, prime subgroups (which are precisely the finitely meet-irreducible elements) onto prime subgroups, regular subgroups (which are the meet-irreducible elements) onto regular subgroups, and polars (which are the skeletal elements) onto polars. μ also maps cardinal summands onto cardinal summands, since P is a cardinal summand of G if and only if P is a polar and $G = P \vee P^{\perp}$ in $\mathbf{C}(G)$.

Other convex l-subgroups do not fare as well. For example, l-ideals may not remain normal in an a-extension. A case in point is to let G be a splitting extension of $\mathbb{R} \boxplus \mathbb{R}$ by $\mathbb{Z}$, where

$$((x_1, y_1), m_1)((x_2, y_2), m_2) = ((x_1 + x_2, y_1 + y_2), m_1 + m_2).$$

An a-extension of G is the following splitting extension H of $\mathbb{R} \boxplus \mathbb{R}$ by $\mathbb{Z} \boxplus \mathbb{Z}\sqrt{2}$, where

$$\begin{aligned}&\left((x_1, y_1), m_1 + n_1\sqrt{2}\right)\left((x_2, y_2), m_2 + n_2\sqrt{2}\right)\\ &\qquad = \begin{cases} \left((x_1 + x_2, y_1 + y_2), m_1 + m_2 + (n_1 + n_2)\sqrt{2}\right), & \text{if } n_1 \text{ is even;} \\ \left((x_1 + y_2, x_2 + y_1), m_1 + m_2 + (n_1 + n_2)\sqrt{2}\right), & \text{if } n_1 \text{ is odd,} \end{cases}\end{aligned}$$

where G is l-embedded naturally into H by

$$((x, y), m) \to \left((x, y), m + 0\sqrt{2}\right).$$

In both G and H, $L = \{((x, y), m + n\sqrt{2}) : y = m = n = 0\}$ is a convex l-subgroup. However, $L \lhd G$ while $L \ntriangleleft H$.

The following proposition is about the best that one can state for an arbitrary convex l-subgroup of an l-group G in an a-extension H.

PROPOSITION 3.4. *Let H be an a-extension of an l-group G and let C be a convex l-subgroup of G. Then $H(C)$ is an a-extension of C. Moreover, if C is a-closed, then $C = H(C)$.*

PROPOSITION 3.5. *Let H be an a-extension of an l-group G and suppose that L is an l-ideal of H such that $L = L \cap G$. If G/L is a-closed, then $H = G$.*

Proof. Let $a < h \in H \setminus L$. Then there exist $e < g \in G$ and positive integers m and n such that $h < g^m$ and $g < h^n$. Then $Lh \leq Lg^m = (Lg)^m$ and $Lg \leq Lh^n = (Lh)^n$. Thus H/L is an a-extension of G/L. Since G/L is a-closed, $H/L = G/L$.

Hence there must exist $a \in G$ and $x \in L$ such that $h = xa$. But since $L \subseteq G$, $xa \in G$. ∎

Surprisingly, not much is known about the convex l-subgroups of an a-closed l-group. One outstanding unsolved question is whether a convex l-subgroup of an a-closed l-group is a-closed. One case in which the answer is known is the following.

PROPOSITION 3.6. *Let G be an a-closed l-group and S be a cardinal summand of G. Then S is a-closed.*

Proof. Let T be an a-extension of S; let $H = T \boxplus S^{\perp}$ and $e < h = tx$, where $e \leq t \in T$ and $e \leq x \in S^{\perp}$. Choose $s \in S$ and positive integers m and n such that $s < t^m$ and $t < s^n$. Then $sx \in G$ and $sx < t^m x^m = (tx)^m$, while $tx < s^n x^n = (sx)^n$. So H is an a-extension of G, and is proper unless $T = S$. ∎

PROPOSITION 3.7. *Let $G = \sum_\Lambda G_\lambda$ be a cardinal sum of l-groups and let H be an a-extension of G. Then $H = \sum_\Lambda H_\lambda$, where for all $\lambda \in \Lambda$, H_λ is an a-extension of G_λ.*

Moreover, if $\sum_\Lambda G_\lambda \subseteq G \subseteq \prod_\Lambda G_\lambda$ and H is an a-extension of G, then for all $\lambda \in \Lambda$, there exists an a-extension H_λ of G_λ such that $\sum_\Lambda H_\lambda \subseteq H \subseteq \prod_\Lambda H_\lambda$.

Proof. For all λ, G_λ is a cardinal summand of G and so $H_\lambda = H(G_\lambda)$ is, by the remarks after Theorem 3.3, a cardinal summand of H. Since H_λ is a cardinal summand, $H_\lambda \triangleleft H$, and so $\sum_\Lambda H_\lambda = \bigvee_{\mathcal{C}(H)} H_\lambda \triangleleft H$.

Now in the event that $G = \sum_\Lambda G_\lambda$, then $G = \bigvee_{\mathcal{C}(G)} G_\lambda$ implies that $H = \bigvee_{\mathcal{C}(H)} H_\lambda = \sum_\Lambda H_\lambda$.

Suppose though that $\sum_\Lambda G_\lambda \subset G \subseteq \prod_\Lambda G_\lambda$. Let $e < h \in H$; choose $e < g = \bigvee_\Lambda g_\lambda \in G$, where $g_\lambda \in G_\Lambda$, and a positive integer m such that $h < g^m$. Then $h = h \wedge (\bigvee_\Lambda g_\lambda)^m = h \wedge \bigvee_\Lambda g_\lambda^m = \bigvee_\Lambda (h \wedge g_\lambda^m)$. Let $h_\lambda = h \wedge g_\lambda^m$. Now for any $\lambda \in \Lambda$ and any $\mu \neq \lambda$, $h_\lambda \wedge h_\mu = e$, and so in the decomposition of H as $H_\lambda \boxplus H_\lambda^{\perp}$, $h = h_\lambda x$, where $x \in H_\lambda^{\perp}$. Thus h_λ is independent of the choice of g.

We can thus define $\sigma: H \to \prod_\Lambda H_\lambda: h \to (\ldots, h_\lambda, \ldots)$. σ is clearly an l-isomorphism. ∎

COROLLARY 3.8. *The cardinal sum and cardinal product of a-closed l-groups are a-closed.*

PROPOSITION 3.9. *For each $\lambda \in \Lambda$, let H_λ be an a-closed l-group and K be an a-closed l-group such that $\Sigma_\Lambda H_\lambda \subseteq K \subseteq \prod_\Lambda H_\lambda$. Then $K/(\Sigma_\Lambda H_\lambda)$ is a-closed.*

Proof. That $\Sigma_\Lambda H_\lambda \lhd K$ appears in the proof of Proposition 3.7. Suppose that $L \supseteq K$ is an l-group such that $L/(\Sigma_\Lambda H_\lambda)$ is an a-extension of $K/(\Sigma_\Lambda H_\lambda)$.

First of all, choose $\lambda_0 \in \Lambda$ and suppose that $e < x \in (\Sigma_{\mu \neq \lambda_0} H_\mu)^\perp$ in L. There exists $k \in K$ such that $(\Sigma_\Lambda H_\lambda)x$ is a-equivalent to $(\Sigma_\Lambda H_\lambda)k$, and so there exist $s \in \Sigma_\Lambda H_\lambda$ and a positive integer m such that $x < sk^m$. But then $x = \bigvee_\Lambda (x \wedge s_\lambda k_\lambda^m)$, and for all $\mu \neq \lambda_0$, $x \wedge s_\mu k_\mu^m = e$. Thus $x = x \wedge s_{\lambda_0} k_{\lambda_0}^m \in H_{\lambda_0}$ and so H_{λ_0} is a polar in L. Moreover, since for any $e \leq y \in L$ and $k' \in K$, $t \in \Sigma_\Lambda H_\lambda$, and positive integer n such that $y < t(k')^n$,

$$y = \bigvee_\Lambda \left(y \wedge t_\lambda (k')_\lambda^n\right) = \left(y \wedge t_{\lambda_0}(k')_{\lambda_0}^n\right) \vee \bigvee_{\mu \neq \lambda_0} \left(y \wedge t_\mu (k')_\mu^n\right)$$

shows that $L = H_{\lambda_0} \vee (H_{\lambda_0})^\perp$, and so H_{λ_0} is a cardinal summand of L.

Again, let $e < x \in L$; there exist $k \in K$, $a, b \in \Sigma_\Lambda H_\lambda$, and positive integers m and n such that $k < ax^m$ and $x < bk^n$. Let Λ_0 be the set of $\lambda \in \Lambda$ such that either $a_\lambda \neq e$ or $b_\lambda \neq e$; then Λ_0 is finite and so $\Sigma_{\Lambda_0} H_\lambda$ is a cardinal summand of L. Let r and s be the projections of x and k, respectively, onto Λ/Λ_0; then r is a-equivalent to $s \in K$. Since $\sigma_{\Lambda_0} H_\lambda$ is also a cardinal summand of K and $\Sigma_{\Lambda_0} H_\lambda$ is a-closed, the projection of x onto $\Sigma_{\Lambda_0} H_\lambda$ is in K. Since K is a-closed, $x \in K$, and so $L = K$. ∎

In [15], Conrad and Darnel defined the *eventually constant extension* G^{EC} of an l-group G to be the l-subgroup of $\prod_{i=1}^\infty G$ generated by $\Sigma_{i=1}^\infty G$ and the "long constants": for $g \in G$, $\bar{g}_i = g$ for all $1 \leq i < \infty$.

PROPOSITION 3.10. *Let H be an a-closure of an l-group G. Then H^{EC} is an a-closure of G^{EC}.*

Proof. First of all, let $e < x \in H^{\mathrm{EC}}$. If $x \in \Sigma_{i=1}^\infty H$, then since by Proposition 3.7, $\Sigma_{i=1}^\infty H$ is an a-extension of $\Sigma_{i=1}^\infty G$, there clearly exists $g \in \Sigma_{i=1}^\infty G$ that is a-equivalent to x. Otherwise, there exists $e < h \in H$ such that for all but finitely many i, $x_i = h$. Choose $e < g \in G$ such that g is a-equivalent to h, and let b be the element of G^{EC} such that if $x_i = h$, $b_i = g$, while if $x_i \neq h$, $b_i = 0$. Let $I = \{i\colon x_i \neq h\}$. Define $y \in H^{\mathrm{EC}}$ by

$$y_i = \begin{cases} 0, & \text{if } i \notin I; \\ x_i, & \text{if } i \in I. \end{cases}$$

Then $e \leq y \in \sum_{i=1}^{\infty} H$; choose $e \leq a \in \sum_{i=1}^{\infty} G$ such that a is a-equivalent to y. Then $ab \in G^{\mathrm{EC}}$ and ab is a-equivalent to x. So H^{EC} is an a-extension of G^{EC}.

Now let K be an a-extension of H^{EC}. By Proposition 3.7, $\sum_{i=1}^{\infty} H \subset H^{\mathrm{EC}} \subseteq K \subseteq \prod_{i=1}^{\infty} H$ and, as stated in the proof of Proposition 3.7, $\sum_{i=1}^{\infty} H \triangleleft K$. However, $H^{\mathrm{EC}}/\sum_{i=1}^{\infty} H \cong H$, and so by Proposition 3.5, $K = H^{\mathrm{EC}}$. ▮

COROLLARY 3.11. *If G is a-closed, then G^{EC} is a-closed.*

Now suppose that G is an l-group having a unique a-closure H. By Proposition 3.8, H^{EC} is an a-closure of G^{EC}. However, it is not known if H^{EC} is the unique a-closure of G^{EC}. The following is the best possible at this time.

PROPOSITION 3.12. *Let G be an l-group having a unique a-closure H. Then for any a-closure K of G^{EC}, $\sum_{i=1}^{\infty} H \triangleleft K$ and $K/\sum_{i=1}^{\infty} H \cong H$.*

Proof. By Proposition 3.9, $K/(\sum_{i=1}^{\infty} H)$ is a-closed. It is easily verified that $K/(\sum_{i=1}^{\infty} H)$ is an a-extension of

$$\frac{G^{\mathrm{EC}}(\sum_{i=1}^{\infty} H)}{(\sum_{i=1}^{\infty} H)} \cong \frac{G^{\mathrm{EC}}}{G^{\mathrm{EC}} \cap \sum_{i=1}^{\infty} H} = \frac{G^{\mathrm{EC}}}{\sum_{i=1}^{\infty} G},$$

and this is l-isomorphism to G. So $K/(\sum_{i=1}^{\infty} H) \cong H$. ▮

In [2], Anderson and Conrad proved a remarkable theorem concerning a-extensions. Their proof of the theorem actually proves a more general result, which we state and prove here.

THEOREM 3.13. *Let H be an a-extension of an l-group G. If for every minimal prime subgroup M of H and every $h \in H$, there exists $g \in G$ such that $Mh = Mg$, then $G = H$.*

Proof. Suppose there exists $e < h \in H \setminus G$. Let $\mathcal{F} = \{hg^{-1} : g \in G$ and $g < h\}$. If $g_1, g_2 \in G$ such that $hg_1^{-1}, hg_2^{-1} \in \mathcal{F}$, then $hg_1^{-1} \wedge hg_2^{-1} = h(g_1^{-1} \wedge g_2^{-1}) = h(g_1 \vee g_2)^{-1} > e$. So $\mathcal{F}$ is a filter in H^+ and so is contained in an ultrafilter $\mathcal{U}$ on H^+. Then $H^+ \setminus \mathcal{U}$ is the positive cone of a minimal prime subgroup M of H.

By assumption, there exists $g \in G$ such that $Mh = Mg$. $hg^{-1} \in M$ implies that $x = hg^{-1} \vee e$ and $y = gh^{-1} \vee e$ are in M. Let $z \in G$ such that there exists an integer n such that $y < z^n$ and $z < y^n$; then $z \in M$ as well. So $Mz^{-n}g = Mg = Mh$.

However, $hg^{-1}z^n = xy^{-1}z^n > e$. Thus $hg^{-1}z^n \in \mathcal{F}$ and since $M^+ \subseteq H^+ \setminus \mathcal{F}$, $hg^{-1}z^n \notin M$. On the other hand, since $Mh = Mg$ and $z \in M$, $hg^{-1}z^n \in M$. ▮

Anderson and Conrad used Theorem 3.13 to prove many results about a-closed abelian l-groups. However, the theorem is also useful for proving theorems about a-closed nonabelian l-groups.

LEMMA 3.14. *Let H be an a-extension of an l-group G such that there exists $G_\gamma \in \Gamma(G)$ with $G_\gamma \triangleleft G^\gamma$ and $G^\gamma/G_\gamma \cong \mathbb{R}$. Then $H_\gamma \triangleleft H^\gamma$, $H^\gamma/H_\gamma \cong \mathbb{R}$, and for every $h \in H^\gamma$, there exists $g \in G^\gamma$ such that $H_\gamma h = H_\gamma g$.*

Proof. As pointed out in the remarks after Definition 6, $G_\gamma \triangleleft G^\gamma$ implies $H_\gamma \triangleleft H^\gamma$. Define $\phi\colon G^\gamma/G_\gamma \to H^\gamma/H_\gamma\colon G_\gamma g \to H_\gamma g$. ϕ is easily shown to be well defined and to be an o-homomorphism which is injective. Since $G^\gamma/G_\gamma \cong \mathbb{R}$, ϕ must be onto and so $H^\gamma/H_\gamma \cong \mathbb{R}$.

But since ϕ is onto, then for any $H_\gamma h \in H^\gamma/H_\gamma$, there exists $G_\gamma g \in G^\gamma/G_\gamma$ such that $H_\gamma g = (G_\gamma g)\phi = H_\gamma h$. ∎

THEOREM 3.15. *Let G be a normal-valued l-group such that $\Gamma(G)$ satisfies the descending chain condition and such that for any $G_\gamma \in \Gamma(G)$, $G^\gamma/G_\gamma \cong \mathbb{R}$. Then G is a-closed.*

Proof. Suppose that G is not a-closed and that H is a proper a-extension of G. By Theorem 3.13, there exist a minimal prime subgroup M of H and $h \in H$ such that there does *not* exist $g \in G$ so that $Mh = Mg$.

Clearly $h \notin M$ and so h has a value $H_1 \supseteq M$. By Lemma 3.14, there exists $g_1 \in G$ such that $hg_1^{-1} \in H_1$. Since $hg_1^{-1} \notin M$, hg_1^{-1} has a value $H_2 \supseteq M$ and $H_2 \subset H_1$. Again by Lemma 3.14, there exists $g_2 \in G$ such that $hg_1^{-1}g_2^{-1} \in H_2$. Continue in this way, then, to produce $H_1 \supset H_2 \supset H_3 \supset \cdots$. But this then contradicts the fact that $\Gamma(G)$ (which is order isomorphic to $\Gamma(H)$) satisfies the descending chain condition. ∎

The hypothesis that G be normal-valued is necessary, as the l-group G of order-preserving permutations of $\mathbb{R}$ that are finitely piecewise linear is an l-group such that $\Gamma(G)$ satisfies the descending chain condition [23].

Conrad [12] proved Theorem 3.15 for the case when G is a vector lattice.

The converse of Theorem 3.15—namely that if G is a-closed and $\Gamma(G)$ satisfies the descending chain condition, then for each $G_\gamma \in \Gamma(G)$, $G^\gamma/G_\gamma \cong \mathbb{R}$—is not true, even if $\Gamma(G)$ is finite. For let D be a subgroup of $\mathbb{R}$ such that $\mathbb{R} = \mathbb{Q} \oplus D$ as a group direct sum. Let G be the splitting extension of $\mathbb{R} \boxplus \mathbb{R}$ by $\mathbb{Z} \oplus D$, where

$$((x_1, y_1)(m_1, d_1))((x_2, y_2),(m_2, d_2)) = \begin{cases} ((x_1 + x_2, y_1 + y_2),(m_1 + m_2, d_1 + d_2)), & \text{if } m_1 \text{ is even;} \\ ((x_1 + y_2, x_2 + y_1),(m_1 + m_2, d_1 + d_2)), & \text{if } m_1 \text{ is odd.} \end{cases}$$

G is then a-closed, $\Gamma(G)$ is finite, but $G/(\mathbb{R} \boxplus \mathbb{R}) \not\cong \mathbb{R}$.

Lemma 3.14 also has other applications which produce nonabelian a-closed l-groups.

PROPOSITION 3.16. *Let A be an a-closed l-group. Then $A \operatorname{Wr} \mathbb{R}$ and $A \operatorname{wr} \mathbb{R}$ are a-closed.*

Proof. We will show this only for $G = A \operatorname{Wr} \mathbb{R}$. Let H be an a-extension of G. Now $P = \prod_{\mathbb{R}} A$ is the maximal proper convex l-subgroup of G and since A is a-closed, P is a-closed. So P is also the maximal proper convex l-subgroup of H.

Let $h \in H$. If $h \in P$, then $h \in G$. If $h \notin P$, then by Lemma 3.14, there exists $g \in G$ such that $hg^{-1} = p \in P$, and so $h = pg \in G$. ∎

Suppose that $A = \mathbb{R}$. By iteration and induction, then, for any positive integer n,

$$\operatorname{Wr}^n(\mathbb{R}, \mathbb{R}) = (\cdots ((\mathbb{R} \underbrace{\operatorname{Wr} \mathbb{R}) \operatorname{Wr} \mathbb{R}) \cdots) \operatorname{Wr} \mathbb{R}}_{n \text{ times}}$$

is a-closed. More generally, if α is any ordinal, then we have that $\operatorname{Wr}_\alpha(\mathbb{R}, \mathbb{R})$ (see [17], [4], or [16] for definitions) is a-closed:

THEOREM 3.17. *For any ordinal α, $\operatorname{Wr}_\alpha(\mathbb{R}, \mathbb{R})$ is a-closed.*

Proof. If $\alpha = 0$, then $\operatorname{Wr}_\alpha(\mathbb{R}, \mathbb{R}) = \mathbb{R}$, and so is a-closed. So suppose that $\alpha > 0$ and that for all ordinals $\beta < \alpha$, $\operatorname{Wr}_\beta(\mathbb{R}, \mathbb{R})$ is a-closed.

If $\alpha = \beta + 1$, then $\operatorname{Wr}_\alpha(\mathbb{R}, \mathbb{R}) = (\operatorname{Wr}_\beta(\mathbb{R}, \mathbb{R})) \operatorname{Wr} \mathbb{R}$, and so is a-closed by Proposition 3.17.

So suppose α is a limit ordinal. Then $\operatorname{Wr}_\alpha(\mathbb{R}, \mathbb{R})$ is special-valued [28]. Thus if H is an a-extension of $\operatorname{Wr}_\alpha(\mathbb{R}, \mathbb{R})$, H is special-valued as well.

Let x be a special element in H; then there exists a special element $g \in \operatorname{Wr}_\alpha(\mathbb{R}, \mathbb{R})$ so that x is a-equivalent to g. By the construction of $\operatorname{Wr}_\alpha(\mathbb{R}, \mathbb{R})$, there exists an ordinal $\beta < \alpha$ such that $g \in \prod_{\alpha \setminus \beta} \operatorname{Wr}_\beta(\mathbb{R}, \mathbb{R})$, which is a-closed. So $x \in \prod_{\alpha \setminus \beta} \operatorname{Wr}_\beta(\mathbb{R}, \mathbb{R}) \subset \operatorname{Wr}_\alpha(\mathbb{R}, \mathbb{R})$. So if $e > h \in H$, then all special components of h are in $\operatorname{Wr}_\alpha(\mathbb{R}, \mathbb{R})$. But $\operatorname{Wr}_\alpha(\mathbb{R}, \mathbb{R})$ is laterally complete [28], and so $h \in \operatorname{Wr}_\alpha(\mathbb{R}, \mathbb{R})$. ∎

Far more is known of abelian a-closures of abelian l-groups. One of the foremost results is the following.

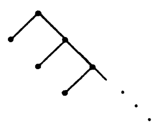
THEOREM 3.18 (Conrad [12]). *If G is abelian, special-valued, divisible, and laterally complete, then $V(\Delta, \mathbb{R})$ is the a-closure of G, where Δ is the minimal plenary subset of $\Gamma(G)$. Thus if G is special-valued, the a-closure of $(G^d)^L = V(\Delta, \mathbb{R})$.*

In the next section, we show that if G is abelian, special-valued, and laterally complete, then $V(\Delta, \mathbb{R})$ is the Δ-closure of G. Thus, in particular, $V(\Delta, \mathbb{R})$ is a-closed.

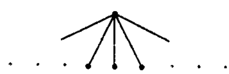
PROPOSITION 3.19. *If G is a completely distributive abelian l-group and Δ is the minimal plenary subset of $\Gamma(G)$, then the a-closure of $(G^d)^L = V(\Delta, \mathbb{R})$.*

Proof. Bixler and Darnel [7] showed that $(G^d)^L$ is special-valued. ∎

DEFINITION 7. Λ_0 will denote the root system



while Λ_1 will denote the root system



DEFINITION 8. A *finite-valued* subgroup of an l-group G is an l-subgroup A such that each $g \in A$ has only finitely many values in G.

PROPOSITION 3.20. *Let G be an abelian finite-valued l-group, $\Delta = \Gamma(G)$, and H be an abelian a-closure of G. Then H is a maximal finite-valued l-subgroup of $V(\Delta, \mathbb{R})$.*

Proof. Since H is an a-closure of G, $\Gamma(H) = \Delta$, and so the v-embedding of G into $V(\Delta, \mathbb{R})$ can be lifted to a v-embedding of H into $V(\Delta, \mathbb{R})$.

Let $v \in V(\Delta, \mathbb{R}) \setminus H$; then the l-subgroup A of $V(\Delta, \mathbb{R})$ generated by $H \cup \{v\}$ is not an a-extension of G, and so there exists an element $0 < a \in A$ which is not a-equivalent to an element of G. Now if a has only finitely many values, then $a = a_1 \vee \cdots \vee a_n$ as a join of disjoint special elements. For each a_i, there exists a special $g_i \in G$ such that $\Gamma_{g_i} = \Gamma_{a_i}$, and so $g = g_1 \vee \cdots \vee g_n$ is a-equivalent to a, which contradicts the fact that a is not a-equivalent to any element of G. So a must have infinitely many values and so H is a maximal finite-valued subgroup of $V(\Delta, \mathbb{R})$. ∎

The converse is also true.

LEMMA 3.21 (Chen, Conrad, and Darnel [9]). *If U is a maximal finite-valued subgroup of $V(\Delta, \mathbb{R})$, there there exists an l-automorphism σ of $V(\Delta, \mathbb{R})$ such that $\Sigma(\Delta, \mathbb{R}) \subseteq U\sigma$ and $U\sigma$ is an a-closure of $\Sigma(\Delta, \mathbb{R})$.*

PROPOSITION 3.22. *For a root system Δ, Δ contains no copy of Λ_1 if and only if $F(\Delta, \mathbb{R}) = \{v \in V(\Delta, \mathbb{R}) : v$ has only finitely many values$\}$. If this is the case, then $F(\Delta, \mathbb{R})$ is a-closed.*

Proof. Suppose that Δ contains no copy of Λ_1. Let $0 < v \in V(\Delta, \mathbb{R})$ such that v has only finitely many values. Then $v = v_1 \vee v_2 \vee \cdots \vee v_n$, where if $i \neq j$, $v_i \wedge v_j = 0$ and each v_i is special. Since Δ contains no copy of Λ_1, each $v_i \in F(\Delta, \mathbb{R})$ and so $v \in F(\Delta, \mathbb{R})$.

Conversely, suppose that $F(\Delta, \mathbb{R}) = \{v \in V(\Delta, \mathbb{R}) : v$ has only finitely many values$\}$ and that Δ contains a copy of Δ_1. Then the characteristic function χ of Λ_1 is in $V(\Delta, \mathbb{R})$ and has exactly one value; thus $\chi \in F(\Delta, \mathbb{R})$, which contradicts the definition of $F(\Delta, \mathbb{R})$.

$F(\Delta, \mathbb{R})$ is thus the maximal finite-valued l-subgroup of $V(D, \mathbb{R})$ and so by Lemma 3.21 is a-closed. ∎

COROLLARY 3.23. *If G is an abelian l-group satisfying Property (F), then $F(\Delta, \mathbb{R})$ is the a-closure of G.*

Proof. Since G satisfies Property (F), G is finite-valued [14], and clearly $\Delta = \Gamma(G)$ contains no copy of Λ_1. So $F(\Delta, \mathbb{R})$ is a-closed and is the unique maximal a-closed finite-valued subgroup of $V(\Delta, \mathbb{R})$.

Since G is finite-valued, we can assume without loss of generality that $\Sigma(\Delta, \mathbb{Z}) \subseteq G$, and since $F(\Delta, \mathbb{R})$ is the a-closure of $\Sigma(\Delta, \mathbb{Z})$, $F(\Delta, \mathbb{R})$ is the a-closure of G. ∎

Many conditions are known to be equivalent to the condition that Δ contains no copy of Λ_1. Among these are that $F(\Delta, \mathbb{R})$ satisfies Property (F), that $F(\Delta, \mathbb{R})$ is convex in $V(\Delta, \mathbb{R})$, that $F(\Delta, \mathbb{R})$ is a characteristic subgroup of $V(\Delta, \mathbb{R})$, and that for any scalar multiplication on $V(\Delta, \mathbb{R})$, $F(\Delta, \mathbb{R})$ is an l-subspace.

PROPOSITION 3.24. *For a root system Δ, the following are equivalent*:

(a) *Δ satisfies the descending chain condition.*

(b) $\Sigma(\Delta, \mathbb{R}) = F(\Delta, \mathbb{R})$.

(c) $\Sigma(\Delta, \mathbb{R})$ *is a-closed.*

(d) $V(\Delta, \mathbb{R})$ *is the lateral completion of* $\Sigma(\Delta, \mathbb{R})$.

(e) $\Sigma(\Delta, \mathbb{R})$ *is a maximal finite-valued l-subgroup of* $V(\Delta, \mathbb{R})$.

(f) *Every maximal finite-valued l-subgroup of* $V(\Delta, \mathbb{R})$ *is l-isomorphic to* $\Sigma(\Delta, \mathbb{R})$.

Proof. That (a) ⇔ (b) is obvious, and (a) ⇔ (c) is now obvious by Proposition 3.16. That (a) ⇔ (d) is obvious, and Lemma 3.22 makes $(b) \Leftrightarrow (c)$ *and* $(e) \Leftrightarrow (f)$ *obvious.* ∎

Now consider Λ_0. Byrd [8] proved that $\Sigma(\Lambda_0, \mathbb{R})$ has an uncountable number of abelian a-closures, all isomorphic as groups to $F(\Lambda_0, \mathbb{R})$ but not l-isomorphic to one another. An extension of Byrd's result is the following.

THEOREM 3.25 (Chen, Conrad, Lin, and Nelson [10]). *An abelian finite-valued l-group G has a unique abelian a-closure if and only if* $\Delta = \Gamma(G)$ *contains no copy of* Λ_0. *If this is the case, then* $F(\Delta, \mathbb{R})$ *is the unique a-closure of G and each l-subgroup of G has a unique a-closure as well.*

In Proposition 3.22, Corollary 3.23, and Theorem 3.25, conditions on Δ were given for which $F(\Delta, \mathbb{R})$ is a-closed. However, in [12], Conrad first gave a long and difficult proof that for any root system Δ, $F(\Delta, \mathbb{R})$ is a-closed. The following is an easier proof based on Lemma 3.21.

PROPOSITION 3.26. *For any root system* Δ, $F(\Delta, \mathbb{R})$ *is a maximal finite-valued subgroup of* $V(\Delta, \mathbb{R})$, *and so is a-closed.*

Proof. Suppose that $F(\Delta, \mathbb{R}) \subset H \subseteq V(\Delta, \mathbb{R})$ and $h \in H \setminus F(\Delta, \mathbb{R})$. Let $\mathcal{M}$ be the set of all maximal infinite antichains in $\operatorname{supp}(h)$, partially ordered by the order inherited from T_Δ. Let $\mathcal{C} = \{\dots, M_\alpha, \dots\}$ be a chain in $\mathcal{M}$ and let M be the set of all maximal elements of $\bigcup_{\mathcal{C}} M_\alpha$.

For $\delta \in M$, there exists $M_\alpha \in \mathcal{C}$ such that $\delta \in M_\alpha$. Moreover, for any $M_\beta \in \mathcal{C}$ such that $M_\alpha \leq M_\beta$, $\delta \in M_\beta$. Thus if M is finite, there exists $M_\alpha \in \mathcal{C}$ such that $M \subseteq M_\alpha \leq M$, and so $M = M_\alpha$. This contradicts the fact that M_α is infinite, and so $M \in \mathcal{M}$. Thus by Zorn's lemma, $\mathcal{M}$ has maximal elements. Let N be a maximal element in $\mathcal{M}$ and $\Lambda = \{\lambda \in \operatorname{supp}(h) : \lambda > \delta \in N\}$. Then $h|_\Lambda \in F(\Delta, \mathbb{R})$ and since N is the set of maximal elements in the support of $h - h|_\Lambda$, $h - h|_\Lambda$ is not finite-valued, and thus H is not a finite-valued subgroup of $V(\Delta, \mathbb{R})$. ∎

An easy alternative method for proving $F(\Delta, \mathbb{R})$ a-closed is to use the following proposition (which is a corollary of Theorem 3.13) from [2].

PROPOSITION 3.27 (Anderson and Conrad [2]). *Let G be an abelian l-group. If G/P is a-closed for each minimal prime subgroup P of G, then G is a-closed.*

COROLLARY 3.28. *For any root system* Δ, $F(\Delta, \mathbb{R})$ *is a-closed. Thus for any root system* Δ, $F(\Delta, \mathbb{R})$ *is a maximal finite-valued subgroup of* $V(\Delta, \mathbb{R})$ *and an a-closure of* $\Sigma(\Delta, \mathbb{R})$.

Proof. Let P be a minimal prime subgroup of $F(\Delta, \mathbb{R})$. Then $F(\Delta, \mathbb{R})/P$ is isomorphic to the projection of $F(\Delta, \mathbb{R})$ onto a root of Δ. So $F(\Delta, \mathbb{R})/P$ is a Hahn group and hence a-closed. ∎

COROLLARY 3.29 [2]. *Each hyperarchimedean vector lattice is a-closed.*

COROLLARY 3.30 [1]. *For a vector lattice G, the following are equivalent*:

(a) $\Gamma(G)$ *satisfies the descending chain condition.*

(b) *Each prime subgroup of G is regular.*

(c) *Each l-subspace is a-closed.*

(d) *G has no proper a-subspaces.*

4. Δ-EXTENSIONS AND Δ-CLOSURES

Suppose that S is an l-subgroup of an l-group G and Δ is a plenary subset of $\Gamma(G)$. Then $\Delta_S = \{\Delta_s : s \in S\}$ is a sublattice of $\Delta_G = \{\Delta_g : g \in G\}$.

DEFINITION 9. Let S be an l-subgroup of an l-group G and let Δ be a plenary subset of $\Gamma(G)$. G is a Δ-*extension* of S if $\Delta_S = \Delta_G$ and S is a Δ-*subgroup* of G.

If in addition, G admits no proper Δ-extension, then G will be said to be Δ-*closed* and to be a Δ-*closure* of S.

PROPOSITION 4.1. *A finite-valued l-group G is* Δ-*closed if and only if G is a-closed.*

Proof. An l-group G is finite-valued if and only if $\Gamma(G)$ has no proper plenary subsets. ∎

Recall that for an l-group G, $C \in \mathcal{C}(G)$, and a plenary subset Δ of $\Gamma(G)$, C is Δ-*convex* if $\Delta_x \leq \Delta_c$ for $c \in C$ implies $x \in C$. $\mathcal{C}_\Delta(G)$ will denote the set of Δ-convex l-subgroups of G.

PROPOSITION 4.2. *If G is a* Δ-*extension of S, then there exist natural isomorphisms between* Δ_S *and* Δ_G, *and between* $\mathcal{C}_\Delta(S)$ *and* $\mathcal{C}_\Delta(G)$.

PROPOSITION 4.3. *If G is a* Δ-*extension of S, then the map* τ: $\mathcal{P}(G) \to \mathcal{P}(S) : Q \to Q \cap S$ *is a Boolean isomorphism of* $\mathcal{P}(G)$ *onto* $\mathcal{P}(S)$.

Proof. Let $^\perp$ denote the polar operation in S and $*$ denote the polar operation in G. To prove the proposition, it suffices to show that for any $P \in \mathbf{P}(S)$, $(P \boxplus P^\perp)^* = (e)$ and that for any $Q \in \mathbf{P}(G)$, $Q \cap S = (Q^* \cap S)^\perp$ [14, Props. 17.3, 17.5].

Suppose by way of contradiction that there exists $e < g \in (P \boxplus P^\perp)^*$. Since G is a Δ-extension of S, there exists $e < s \in S$ such that $\Delta_s = \Delta_g$. By Proposition 2.6, $s \in (P \boxplus P^\perp)^*$, and so is in $(P \boxplus P^\perp)^\perp$. But then $s \in P \cap P^\perp = (e)$. So $(P \boxplus P^\perp)^* = (e)$.

Now let $e \leq t \in (Q^* \cap S)^\perp$ and suppose $t \notin Q$. There exists $g \in Q^*$ such that $t \wedge g > e$, and there exists $e < s \in S$ such that $\Delta_s = \Delta_{t \wedge g}$. Thus $s \in Q^*$ and so $s \wedge t = e$. But $\Delta_{s \wedge t} = \Delta_s \wedge \Delta_t = \Delta_{t \wedge g} \wedge \Delta_t = \Delta_s$. So $s \wedge t > e$. ∎

PROPOSITION 4.4. *Let G be an l-group and $\Delta \subset \Lambda$ be plenary subsets of $\Gamma(G)$. If S is an l-subgroup of G such that G is a Λ-extension of S, then G is a Δ-extension of S. Thus if G is Δ-closed, G is Λ-closed.*

Proof. For any $e < g \in G$, there exists $e < s \in S$ such that $\Lambda_g = \Lambda_s$. But then $\Delta_g = \Delta_s$. ∎

COROLLARY 4.5. *Let G be an l-group such that $\Gamma(G)$ has a minimal plenary subset Δ. If G is Δ-closed, then for any plenary subset Λ of $\Gamma(G)$, G is Λ-closed.*

Proof. Since Δ is the minimal plenary subset, $\Delta \subseteq \Lambda$ for any plenary subset Λ. ∎

COROLLARY 4.6. *If G is an a-extension of S, then for any plenary subset Δ of $\Gamma(G)$, G, is a Δ-extension of S.*

There do exist l-groups, however, which are a-closed but for some plenary subset Δ, are not Δ-closed. An example is $\mathbb{R}^{\mathrm{EC}}$. Since $\mathbb{R}$ is a-closed, so is $\mathbb{R}^{\mathrm{EC}}$. Now $\Gamma(\mathbb{R}^{\mathrm{EC}})$ is trivially ordered, with the regular subgroups being either, for $1 \leq i < \infty$, $M_i = \{h \in \mathbb{R}^{\mathrm{EC}} : h(i) = 0\}$ or $\sum_{i=1}^{\infty} \mathbb{R}$. The minimal plenary subset of $\Gamma(\mathbb{R}^{\mathrm{EC}})$ is $\Delta = \{M_i : 1 \leq i < \infty\}$. But $\mathbb{R}^{\mathrm{EC}}$ is not Δ-closed. Let H be the l-subgroup of $\prod_{i=1}^{\infty} \mathbb{R}$ generated by the long vectors $(x, x, x, \ldots)$ and $(y, 2y, 3y, \ldots)$, where $x, y \in \mathbb{R}$. Then H is a proper Δ-extension of $\mathbb{R}^{\mathrm{EC}}$.

Much of our interest in this area is devoted to the case when G is a completely distributive normal-valued l-group and Δ is the minimal plenary subset of essential values; of especial interest is the case when G is special-valued. The next proposition, in the more general setting of normal-valued l-groups, provides background for results in these cases.

PROPOSITION 4.7. *Let G be an l-group and Δ be a plenary subset of $\Gamma(G)$ consisting of only normal values. Let S be an Δ-subgroup of G. Then*

$\Delta(S) = \{S \cap G_\delta : G_\delta \in \Delta\}$ *is a plenary subset of* $\Gamma(S)$ *containing only normal values and the map* $\rho\colon \Delta \to \Delta(S) : G_\delta \to S \cap G_\delta$ *is a bijection.*

If this is the case, then there exist natural order isomorphisms such that

$$\Delta \cong \Delta(S), \qquad \Delta_G \cong \Delta_S, \qquad \mathcal{C}_\Delta(G) \cong \mathcal{C}_\Delta(S).$$

Proof. We first remark that if Δ consists of only normal values, then G is normal-valued and so S is normal-valued.

Let $G_\delta \in \Delta$; there exists $e < s \in S$ such that $G_\delta \in \Delta_s$. So $s \notin G_\delta \cap S$, while $s \in G^\delta \cap S$. Since $G_\delta \lhd G^\delta$, $G_\delta \cap S \lhd G^\delta \cap S$ and

$$\frac{G^\delta \cap S}{G_\delta \cap S} \cong \frac{G_\delta(G^\delta \cap S)}{G_\delta} \subseteq \frac{G^\delta}{G_\delta}$$

shows that $G^\delta \cap S$ covers $G_\delta \cap S$. Since $G_\delta \cap S$ is a prime subgroup of S, $G_\delta \cap S \in \Gamma(S)$.

Now suppose that G_α, G_β are distinct elements of Δ. Suppose first that $G_\alpha \subset G_\beta$ (or vice versa). Let $s \in S$ such that $G_\alpha \in \Gamma_s$ and let $t \in S$ such that $G_\beta \in \Gamma_t$. Then $s \in G^\alpha \cap S \subseteq G_\beta \cap S$ while $t \notin G_\beta \cap S$ shows that $G_\alpha \cap S \neq G_\beta \cap S$. The remaining case is to consider when G_α is incomparable to G_β. In this case, there exist $e < x \in G^\alpha \setminus G_\alpha$ and $e < y \in G^\beta \setminus G_\beta$ such that $x \wedge y = e$. Since S is a Δ-subgroup of G, there exist $e < s, t \in S$ such that $\Delta_x = \Delta_s$ and $\Delta_y = \Delta_t$. So $s \wedge t = e$. Since $s \in (G_\beta \cap S) \setminus G_\alpha$, ρ must be injective.

The rest of the proof is now immediate. ∎

We remark here that most of the above theorem remains true in the case in which G is not normal-valued. For any $G_\delta \in \Delta$, $G_\delta \cap S$ is prime in S and the map $G_\delta \to G_\delta \cap S$ remains injective. However, it is not known whether or not $G_\delta \cap S$ must always be regular in S.

Recall that a regular subgroup $G_\beta \in \Gamma(G)$ is *essential* if there exists $g \in G$ such that $\Gamma_g \leq G_\beta$, and G_β is *special* if there exists $g \in G$ such that $\Gamma_g = G_\beta$. In general, a special value is always an essential value though an essential value may not be special; they are the same if G is archimedean. Also recall that for any plenary subset Δ of $\Gamma(G)$, $g \in G$ has only finitely many values in Δ implies these are the only values of g in $\Gamma(G)$, and if every value of g is contained in an element $\Delta_\delta \in \Delta$, then every value of g in $\Gamma(G)$ is contained in G_δ. Thus $G_\delta \in \Delta$ is special if and only if there exists $g \in G$ such that $\Delta_g = G_\delta$, and G_δ is essential if and only if there exists $g \in G$ such that $\Delta_g \leq G_\delta$.

PROPOSITION 4.8. *Let Δ be a plenary subset of $\Gamma(G)$.*

(a) *Δ is the minimal plenary subset of $\Gamma(G)$ if and only if each $G_\delta \in \Delta$ is essential, which is if and only if for each $G_\delta \in \Delta$, there exists $g \in G$ such that $\Delta_g \leq G_\delta$.*

(b) *Δ is the set of special values of $\Gamma(G)$ if and only if $\Delta \subseteq \Delta_G$.*

(c) *G is finite-valued with Δ the set of special values if and only if $\Delta_G = \{t \in T_\Delta : t \text{ finite}\}$. In this case, $\Delta = \Gamma(G)$, and so a Δ-closure is an a-closure, which may not be unique.*

(d) *$\Delta_G = T_\Delta$ if and only if Δ has only a finite number of roots.*

PROPOSITION 4.9. *Let Δ be a plenary subset of $\Gamma(G)$ and S be a Δ-subgroup of G.*

(a) *$G_\delta \in \Delta$ is essential or special if and only if $G_\delta \cap S$ is respectively essential or special in S.*

(b) *If G is normal-valued, then S is completely distributive or special-valued if and only if G is respectively completely distributive or special-valued.*

Recall [5] that an l-subgroup S of an l-group G is *saturated* if $x \vee y \in S$ and $x \wedge y = e$, then $x, y \in S$. An l-subgroup S is *weakly saturated* if S contains all special components of its elements.

PROPOSITION 4.10. *Let Δ be a plenary subset of $\Gamma(G)$ and S be a Δ-subgroup of G. Then S is weakly saturated.*

Proof. Let $e < s \in S$ be such that s has special component g. Let G_γ be the value of g. Since G_γ is essential, $G_\gamma \in \Delta$ and $\Delta_g = \{G_\gamma\}$. Since S is a Δ-subgroup of G, there exists $e < t \in S$ such that $\Delta_t = \Delta_g$, and so t is special with value G_γ. There exists an integer n such that $t_n > g$, and so $t^n \wedge s = t^n \wedge (g \vee sg^{-1}) = (t^n \wedge g) \vee (t^n \wedge sg^{-1}) = g$. Thus $g \in S$. ∎

However, even in the case in which G is special-valued and Δ is the plenary subset of special values, a Δ-subgroup S may not be saturated.

In $\prod_{i=1}^\infty \mathbb{R}$, let $T = \sum_{i=1}^\infty \mathbb{R} \oplus \mathbb{R}(1,1,1,\dots) \oplus \mathbb{R}(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$. Let A be the projection of T onto the set of odd indices, B be the projection of T onto the set of even indices, and $G = A \boxplus B$.

Let

$$S = \sum_{i=1}^{\infty} \mathbb{R} \oplus \mathbb{R}(1,1,1,\dots) \oplus \mathbb{R}\left(1,0,\tfrac{1}{3},0,\tfrac{1}{5},\dots\right) \oplus \mathbb{R}\left(0,\tfrac{1}{2},0,\tfrac{1}{4},0,\dots\right).$$

A typical element $s \in S$ looks like

$$s = m + x(1,1,1,\dots) + y\left(1,0,\tfrac{1}{3},0,\dots\right) + z\left(0,\tfrac{1}{2},0,\tfrac{1}{4},\dots\right),$$

where $m \in \sum_{i=1}^\infty \mathbb{R}$ and $x, y, z \in \mathbb{R}$. So

$$s = (m_1 + x + y, m_2 + x + z/2, m_3 + x + y/3, m_4 + x + z/4, \dots).$$

Since there exists an integer N such that for all $i \geq N$, $m_i = 0$ and $|x| > |y|/i$, $|x| > |z|/i$, S is an l-subgroup of G.

The minimal plenary subset Δ of $\Gamma(G)$ is the set $\{M_i : M_i = \{g \in G : g(i) = 0\}\}$. It is easily verified that S is a Δ-subgroup of G. But $(1, 0, 1, 0, \dots) + (0, 1, 0, 1, \dots) = (1, 1, 1, 1, \dots) \in S$, while neither $(1, 0, 1, 0, \dots)$ nor $((0, 1, 0, 1, \dots)$ are in S.

PROPOSITION 4.11. *Let G be an abelian l-group and Δ be a plenary subset of $\Gamma(G)$. Then G has an abelian Δ-closure H such that $G \subseteq H \subseteq V(\Delta, \mathbb{R})$.*

In particular, $V(\Delta, \mathbb{R})$ is Λ-closed for any plenary subset Λ of $\Gamma(V(\Delta, \mathbb{R}))$.

Proof. If T is an abelian Δ-extension of G, then Δ is a plenary subset of $\Gamma(T)$ and so any v-embedding of G into $V(\Delta, \mathbb{R})$ can be extended to a v-embedding of T into $V(\Delta, \mathbb{R})$. Thus we will assume that $G \subseteq T \subseteq V(\Delta, \mathbb{R})$.

Let A be the set of l-subgroups T of $V(\Delta, \mathbb{R})$ such that T is a Δ-extension of G. $\mathsf{A} \neq \varnothing$ since $G \in \mathsf{A}$; partially order A by inclusion. Let C be a chain in A. Then $\bigcup_{\mathsf{C}} C$ is an l-subgroup of $V(\Delta, \mathbb{R})$. For any $0 < v \in \bigcup_{\mathsf{C}} C$, since there exists $C \in \mathsf{C}$ such that $v \in C$, there exists $g \in G$ such that $\Delta_v = \Delta_g$, and so $\bigcup_{\mathsf{C}} C \in \mathsf{A}$. Thus A has maximal elements. Let H be such a maximal element. Then H is an abelian Δ-closure of G. ∎

Now if G is an abelian a-closed l-group, then G is known to be a-closed in the category of all l-groups. The proof of this is strongly dependent on the fact that any a-extension of a normal-valued l-group is normal-valued. At this time, it is not known if, for a proper plenary subset Δ of $\Gamma(G)$, where G is a normal-valued l-group, a Δ-extension must be normal-valued. Thus if G is abelian and Δ-closed for some proper plenary subset Δ, it is possible that G may have some non-normal-valued proper Δ-extension. However, in this case, G cannot have proper normal-valued Δ-extensions.

THEOREM 4.12. *Let G be an abelian l-group, and Δ be a plenary subset of $\Gamma(G)$ such that G is Δ-closed in the category of abelian l-groups. Then G is Δ-closed in the category of normal-valued l-groups.*

Proof. The proof is virtually the same as the proof that if an l-group G is a-closed as an abelian l-group, then G is a-closed. Since this is so, we merely sketch the proof and refer the reader to the proof of Proposition 5.5.3 in [6].

So suppose that G is Δ-closed as an abelian l-group and suppose that H is a nonabelian normal-valued l-group such that H is a proper Δ-extension of G. For each $G_\delta \in \Delta$, let π_δ be an o-isomorphism of G^δ/G_δ into $\mathbb{R}$. We will view G as an l-subgroup of $V(\Delta, \mathbb{R})$.

Let $x \in H \setminus G$ and define $v \in V(\Delta, \mathbb{R})$ as follows. If $G + x \cap V^\delta = \varnothing$, define $v_\delta = 0$. Otherwise, there exist $g \in G$ and $y \in V^\delta$ such that

$g + x = y$; in this case, define $v_\delta = y_\delta - g_\delta$. (As pointed out in [6], this definition of v_δ is independent of the choice of g and y.) v is then in $V(\Delta, \mathbb{R}) \setminus G$; moreover, $\operatorname{supp}(v) = \Delta_x$.

Let K be the l-subgroup of $V(\Delta, \mathbb{R})$ generated by $G \cup \{v\}$. K is then a proper abelian Δ-extension of G, which is a contradiction. ∎

THEOREM 4.13. *Let G be a special-valued abelian l-group and let Δ be the minimal plenary subset of $\Gamma(G)$. If G is Δ-closed as an abelian l-group, then G is Δ-closed in the class of all l-groups.*

Proof. Since G is special-valued, any Δ-extension H of G must be normal-valued. ∎

COROLLARY 4.14. *For any root system Δ and any plenary subset Λ of $\Gamma(V(\Delta, \mathbb{R}))$, $V(\Delta, \mathbb{R})$ is Λ-closed.*

PROPOSITION 4.15. *Let G be an abelian l-group and Δ be a plenary subset of $\Gamma(G)$. The following are equivalent*:

(a) $\Delta_G = T_\Delta$.

(b) $V(\Delta, \mathbb{R})$ *is a Δ-extension of G.*

(c) $V(\Delta, \mathbb{R})$ *is the abelian Δ-closure of G.*

Proof. (a) ⇒ (b) Since $\Delta_G = T_\Delta$, $V(\Delta, \mathbb{R})$ is a Δ-extension of G.

(b) ⇒ (c) If H is an abelian Δ-extension of G, then we can assume without loss of generality that $G \subseteq H \subseteq V(\Delta, \mathbb{R})$. Since we are assuming that $V(\Delta, \mathbb{R})$ is a Δ-extension of G, we must have that $V(\Delta, \mathbb{R})$ is the abelian Δ-closure of G.

(c) ⇒ (a) $\Delta_G = \Delta_{V(\Delta, \mathbb{R})} = T_\Delta$. ∎

COROLLARY 4.16. *Let G be a completely distributive abelian l-group and let Δ be the minimal plenary subset of $\Gamma(G)$. Then $V(\Delta, \mathbb{R})$ is the abelian Δ-closure of G^L, the lateral completion of G.*

Proof. G^L is special-valued [7] and abelian. ∎

COROLLARY 4.17. *Let G be an abelian l-group and Δ be a plenary subset of $\Gamma(G)$ such that Δ has only finitely many roots. Then $F(\Delta, \mathbb{R}) = V(\Delta, \mathbb{R})$ is the abelian Δ-closure of G, and is also the abelian a-closure of G.*

Proof. Since Δ has only finitely many roots, G is finite-valued and so $\Delta = \Gamma(G)$. The rest now follows from Theorem 3.18. ∎

For any arbitrary root system Δ, there are (potentially) many proper l-subgroups of $V(\Delta, \mathbb{R})$ that are Δ-closed. We now show a family of such l-subgroups.

DEFINITION 10. Let G be an l-group, Δ be a plenary subset of $\Gamma(G)$, and κ be an infinite cardinal. An l-subgroup A of G is *κ_Δ-valued* if for each $a \in A$, $|\Delta_a| \le \kappa$. A is *$<\kappa_\Delta$-valued* if for each $a \in A$, $|\Delta_a| < \kappa$.

In [9], it was shown that a maximal finite-valued l-subgroup is saturated, and in [15], it was shown that a maximal countably valued l-subgroup is saturated. These results are easily extensible to showing that for any infinite cardinal κ, a maximal κ-valued l-subgroup or a maximal $<\kappa$-valued l-subgroup is saturated.

LEMMA 4.18. *Let G be an l-group, P a prime subgroup of G, $g \in G$, and x a component of g. Then for any $a_1, a_2, \ldots, a_n \in G$ and exponents $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n = \pm 1$, there exist $y_1, y_2, \ldots, y_n \in \{e, g\}$ such that*

$$Pa_1 x^{\varepsilon_1} a_2 x^{\varepsilon_2} \cdots a_{n-1} x^{\varepsilon_{n-1}} a_n = Pa_1 y_1^{\varepsilon_1} a_2 y_2^{\varepsilon_2} \cdots a_{n-1} y_{n-1}^{\varepsilon_{n-1}} a_n.$$

Proof. For $1 \le i \le n$, let $b_i = a_1 x^{\varepsilon_1} a_2 x^{\varepsilon_2} \cdots a_i$. Define

$$y_i = \begin{cases} e, & \text{if } x \in b_i^{-1} P b_i; \\ g, & \text{if } x \notin b_i^{-1} P b_i. \end{cases}$$

Now if $x \in a_1^{-1} P a_1$, then $a_1 x^{\varepsilon_1} a_1^{-1} \in P$ implies that $Pa_1 x^{\varepsilon_1} a_1^{-1} = P$, and so $Pa_1 x^{\varepsilon_1} = Pa_1 e^{\varepsilon_1} = Pa_1 y_1^{\varepsilon_1}$. If $x \notin a_1^{-1} P a_1$, then $g \notin a_1^{-1} P a_1$. Since $g = xx^{-1} g = xgx^{-1}$ and $|x| \wedge |gx^{-1}| = e$, $x^{-1} g \in a_1^{-1} P a_1$ implies $Pa_1 (x^{-1} g)^{\varepsilon_1} = Pa_1$. Now if $\varepsilon_1 = 1$, then $Pa_1 = Pa_1 (x^{-1} g)^{\varepsilon_1} = Pa_1 (gx^{-1})$ implies $Pa_1 y_1^{\varepsilon_1} = Pa_1 g = Pa_1 x$. If $\varepsilon_1 = -1$, then $Pa_1 = Pa_1 (x^{-1} g)^{-1} = Pa_1 g^{-1} x$ implies that $Pa_1 y_1^{\varepsilon_1} = Pa_1 g^{-1} = Pa_1 x^{-1}$. So in either case, $Pa_1 y_1^{\varepsilon_1} = Pa_1 x^{\varepsilon_1}$. But then $Pa_1 y_1^{\varepsilon_1} a_2 = Pa_1 x^{\varepsilon_1} a_2$, and continuing in this way proves the lemma. ∎

PROPOSITION 4.19. *Let κ be an infinite cardinal and let A be a maximal κ_Δ-valued or $<\kappa_\Delta$-valued l-subgroup of an l-group G, where Δ is a plenary subset of $\Gamma(G)$. Then A is saturated.*

Proof. Let $g \in A$ and x be a component of g. Let B be the l-subgroup of G generated by $A \cup \{x\}$. An element of B is of the form $\bigvee_I \bigvee_J b_{ij}$, where I, J are finite sets, and b_{ij} is of the form

$$b_{ij} = a_1 x^{\varepsilon_1} a_2 x^{\varepsilon_2} \cdots a_{n_{ij}-1} x^{\varepsilon_{n_{ij}-1}} a_{n_{ij}},$$

where $a_i \in A$ and $\varepsilon_i = \pm 1$.

Let $h = \bigvee_I \bigvee_J b_{ij}$ and let $G_\delta \in \Delta_h$. G_δ must be a value of some b_{ij}. By the above lemma for some choice $y_1, y_2, \ldots, y_{n_{ij}-1} \in \{e, g\}$, G_δ must be a value of $a_1 y_1^{\varepsilon} a_2 y_2^{\varepsilon_2} \cdots y_{n_{ij}-1}^{\varepsilon_{n_{ij}-1}}$.

Let $n = \max\{n_{ij} : (i, j) \in I \times J\}$. If A is a maximal κ-valued l-subgroup, then $|\Delta_h| \le |I|\, |J| 2^n \kappa = \kappa$, and so B is κ-valued. So $A = B$ and $x \in A$.

Suppose A is a maximal $<\kappa$-valued l-subgroup. For $(i,j) \in I \times J$ and $f \in \{e, g\}^{n_{ij}-1}$, let c_{ijf} be the element of A produced in the above fashion by letting $y_k = f(k)$. Let $\kappa_1 = \max\{|\Delta_{c_{ijf}}|\}$; then $\kappa_1 < \kappa$, and

$$|\Delta_h| \leq |I|\,|J|\,2^n \kappa_1 < \kappa.$$

So $x \in A$. ∎

PROPOSITION 4.20. *Let Δ be a root system. Let κ be an infinite cardinal and A be a maximal κ_Δ-valued or $<\kappa_\Delta$-valued l-subgroup of $V(\Delta, \mathbb{R})$. Then A is Δ-closed.*

Proof. It suffices to show that A has no proper abelian Δ-extension B. For if so, the fact that A is saturated implies that A contains all special components of its elements, and so A is special-valued. Thus any Δ-extension of A must be normal-valued, and so by Theorem 4.12, A is Δ-closed.

So assume A has a proper abelian Δ-extension B; without loss of generality, we can assume that $A \subset B \subseteq V(\Delta, \mathbb{R})$. So there exists $b \in B$ such that $|\Delta_b| > \kappa$ (or, if A is a maximal $<\kappa_\Delta$-valued l-subgroup, $\geq \kappa$). Thus $\Delta_b \neq \Delta_a$ for any $a \in A$, and so B is not a Δ-extension of A. ∎

A now easy result is the following:

COROLLARY 4.21. *$V(\Delta, \mathbb{R})$ is Δ-closed.*

Proof. Choose $\kappa > |V(\Delta, \mathbb{R})|$. ∎

COROLLARY 4.22. *$F(\Delta, \mathbb{R})$ is a maximal finite-valued l-subgroup and so is Δ-closed.*

Proof. See Proposition 3.26. ∎

Since $F(\Delta, \mathbb{R}) = \{v \in V(\Delta, \mathbb{R}) : \operatorname{supp}(v)$ contains no infinite antichains$\}$, it is reasonable to ask if other l-subgroups generated by similar antichain conditions are either Δ-closed or a-closed.

Specifically, let κ be an infinite cardinal and let $D_\kappa(\Delta, \mathbb{R}) = \{v \in V(\Delta, \mathbb{R}) :$ for any antichain $A \subseteq \operatorname{supp}(v)$, $|A| < \kappa\}$. $D_\kappa(\Delta, \mathbb{R})$ is easily verified to be an l-subgroup of $V(\Delta, \mathbb{R})$. However, even for $\kappa = \omega_1$, $D_\kappa(\Delta, \mathbb{R})$ need not be Δ-closed, as the following example shows.

In [26], a *tree* is defined to be a partially ordered set T such that for every $t \in T$, the set $\{g \in T \mid y < t\}$ is well ordered. For a tree T and $x \in T$, the *height* of x is the ordinal type of $\{y \in T \mid y < x\}$. For an ordinal α, $\operatorname{Lev}_\alpha(T) = \{x \in T \mid \operatorname{height}(x) = \alpha\}$. The *height* of T is the least ordinal α such that $|\operatorname{Lev}_\alpha(T)| = 0$. For a tree T and a regular cardinal κ, T is a *κ-tree* if height $(T) = \kappa$ and for every $\alpha < \kappa$, $|\operatorname{Lev}_\alpha(T)| < \kappa$. A *$\kappa$-Aronszajn tree* is a κ-tree T such that for every chain $C \subseteq T$, $|C| < \kappa$. An ω_1-

Aronszajn tree T is *special* if T is the union of countably many antichains. A κ-*Suslin tree* is a κ-Aronszajn tree T such that for any antichain $A \subseteq T$, $|A| < \kappa$. Clearly a special ω_1-Aronszajn tree is not an ω_1-Suslin tree. ZFC implies the existence of a special ω_1-Aronszajn tree.

Now if T is a tree, then reversing the order on T creates a root system, Δ, in which the analogy of height is *depth*. The above definitions now are obviously reversible to define κ root systems, κ-Aronszajn root systems, and κ-Suslin root systems; moreover, ZFC implies the existence of special ω_1-Aronszajn root systems while the existence of ω_1-Suslin root systems is independent of ZFC.

Let Δ be a special ω_1-Aronszajn root system. Since Δ is not an ω_1-Suslin root system, there exists an uncountable antichain in Δ. Since Δ satisfies the ascending chain condition, the characteristic function x of Δ is in $V(\Delta, \mathbb{R})$, but is not in $D_{\omega_1}(\Delta, \mathbb{R})$. Let B be the l-subgroup of $V(\Delta, \mathbb{R})$ generated by $D_{\omega_1}(\Delta, \mathbb{R})$ and x.

Suppose by way of contradiction that B is not countably Δ-valued. Since finite joins and meets of countably Δ-valued elements are countably Δ-valued, there exist $g \in D_{\omega_1}(\Delta, \mathbb{R})$ and a positive integer n such that $\Delta_{(g+nx)}$ is uncountable. Since $D_{\omega_1}(\Delta, \mathbb{R})$ is divisible, we can assume that $n = 1$.

Let $\Omega = \{\beta \in \Delta :$ there exists $\delta \in \Delta_{(g+x)}$ such that $\beta > \delta\}$. For all $\beta \in \Omega$, $g(\beta) = -1$ and so $\Omega \subseteq \operatorname{supp}(g)$. Thus for any antichain $A \subseteq \Omega$, $|A| < \omega_1$. However, Ω is also a dual ideal of Δ. Thus for any $\beta \in \Omega$, $\{\delta \in \Delta : \delta > \beta\}$ is inversely well ordered. Also, for any chain $C \subseteq \Omega$, $|C| < \omega_1$. For any ordinal $\alpha < \omega_1$, $|\operatorname{Lev}_\alpha(\Omega)| < \omega_1$. Finally, $\operatorname{depth}(\Omega) \leq \operatorname{depth}(\Delta) = \omega_1$. Thus if we can prove that $\operatorname{depth}(\Omega) = \omega_1$, then Ω would be an ω_1-Suslin root system.

For $\delta \in \Delta_{(g+x)}$, let $\alpha_\delta = \operatorname{depth}(\delta)$. Suppose there exists $\alpha < \omega_1$ such that for all $\delta \in \Delta_{(g+x)}$, $\alpha_\delta \leq \alpha$. This would imply that $\Delta_{(g+x)} \subseteq \bigcup_{\beta \leq \alpha} \operatorname{Lev}_\beta(\Delta)$. Since for any $\beta < \omega_1$, $|\operatorname{Lev}_\beta(\Delta)| < \omega_1$, $|\bigcup_{\beta \leq \alpha} \operatorname{Lev}_\beta(\Delta)| < \omega_1$, and so $\Delta_{(g+x)}$ would be countable, which is false. So for any $\alpha < \omega_1$, there exists $\delta_\alpha \in \Delta_{(g+x)}$ such that $\operatorname{depth}(\delta_\alpha) > \alpha$. Thus there exists $\beta > \delta_\alpha$ such that $\operatorname{depth}(\beta) = \alpha$. So for any $\alpha < \omega_1$, there exists $\beta \in \Omega$ such that $\operatorname{depth}(\beta) = \alpha$. Thus $\operatorname{depth}(\Omega) = \omega_1$.

So Ω is an ω_1-Suslin root system. However, since Δ is a special ω_1-Aronszajn root system, there exist antichains $A_1, A_2, A_3, \ldots \subseteq \Delta$ such that $\Delta = \bigcup_{n=1}^\infty A_n$. But then $\Omega = \bigcup_{n=1}^\infty (A_n \cap \Omega)$ is a special ω_1-Aronszajn root system, and so cannot be Suslin.

So B is countably Δ-valued. But then B is a proper Δ-extension of $D_{\omega_1}(\Delta, \mathbb{R})$ and so $D_{\omega_1}(\Delta, \mathbb{R})$ is not Δ-closed.

There do exist root systems Δ and Δ-closed l-subgroups of $V(\Delta, \mathbb{R})$ that, for any cardinal κ, are not maximal κ_Δ-valued l-subgroups. As an example, let Δ be a countably infinite trivially ordered set $\{\delta_1, \delta_2, \ldots\}$. Let $\Delta_1 = \{\delta_i \in \Delta : i \text{ is odd}\}$ and $\Delta_2 = \{\delta_i \in \Delta : i \text{ is even}\}$. Let $A = \Sigma_{\Delta_1} \mathbb{R}$ and $B = V(\Delta_2, \mathbb{R})$. Then $A \boxplus B$ is Δ-closed, but is not a maximal ω_Δ-valued l-subgroup of $V(\Delta, \mathbb{R})$.

On the other hand, with the aid of the following lemma, we can prove that for any cardinal κ, $D_\kappa(\Delta, \mathbb{R})$ is Δ-closed if and only if $D_{\omega_1}(\Delta, \mathbb{R})$ is a-closed.

LEMMA 4.23. *Let G be an l-group and Δ be a plenary subset of $\Gamma(G)$. Let $a, b \in G$ such that $\Delta_a = \Delta_b$ and there exist positive integers m and n so that for any $G_\delta \in \Delta_a$, $G_\delta a < G_\delta b^m$ and $G_\delta b < G_\delta a^n$. Then a is a-equivalent to b.*

Proof. Suppose that a is not a-equivalent to b; we can assume that there exists no positive integer m such that $a < b^m$. But then for any integer m, $ab^{-m} \vee e \neq e$, and so has a value $G_m \in \Delta$. But then $G_m ab^{-m} > G_m$, implying $G_m a > G_m b^m$. This contradicts our assumption and so a is a-equivalent to b. ∎

PROPOSITION 4.24. *Let κ be a cardinal. Then $D_\kappa(\Delta, \mathbb{R})$ is Δ-closed if and only if $D_{\omega_1}(\Delta, \mathbb{R})$ is a-closed.*

Proof. We have already seen that if $D_\kappa(\Delta, \mathbb{R})$ is Δ-closed, then $D_\kappa(\Delta, \mathbb{R})$ is a-closed. So suppose that $D_\kappa(\Delta, \mathbb{R})$ is a-closed, and that H is an abelian Δ-extension of $D_\kappa(\Delta, \mathbb{R})$. Let $0 < h \in H$ and let g be the projection of h onto Δ_h. Since $|\Delta_h| < \kappa$, $g \in D_\kappa(\Delta, \mathbb{R})$. By the above lemma, g is a-equivalent to h, and so H is an a-extension of $D_\kappa(\Delta, \mathbb{R})$. Thus $H = D_\kappa(\Delta, \mathbb{R})$. ∎

From the example given above in which $D_{\omega_1}(\Delta, \mathbb{R})$ was not Δ-closed, we see that $D_{\omega_1}(\Delta, \mathbb{R})$ also need be neither a-closed nor a maximal ω_1-valued l-subgroup of $V(\Delta, \mathbb{R})$.

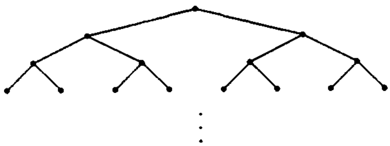
For a cardinal κ, define $C_\kappa(\Delta, \mathbb{R})$ to be the set $\{v \in V(\Delta, \mathbb{R}) : \operatorname{supp}(v)$ is contained in at most κ many roots of $\Delta\}$. It is clear that for any root system Δ and cardinal κ, $C_\kappa(\Delta, \mathbb{R})$ is an l-subgroup of $D_\kappa(\Delta, \mathbb{R})$.

PROPOSITION 4.25. *For any root system Δ and any cardinal κ, $D_\kappa(\Delta, \mathbb{R})$ is an a-extension of $C_\kappa(\Delta, \mathbb{R})$ and so is a Δ-extension of $C_\kappa(\Delta, \mathbb{R})$.*

Proof. For $v \in D_\kappa(\Delta, \mathbb{R})$, let h be its projection onto Δ_v. Then $h \in C_\kappa(\Delta, \mathbb{R})$ and by Lemma 4.23, h is a-equivalent to v. ∎

COROLLARY 4.26. *$C_\kappa(\Delta, \mathbb{R})$ is a-closed if and only if $C_\kappa(\Delta, \mathbb{R}) = D_\kappa(\Delta, \mathbb{R})$ and $D_\kappa(\Delta, \mathbb{R})$ is a-closed.*

We can obtain nicer characterizations those cases in which $C_\kappa(\Delta, \mathbb{R})$ is a-closed when $\kappa = \omega_1$. Let Λ_2 be the root system



THEOREM 4.27. *For a root system Δ, the following are equivalent*:

(a) $C_{\omega_1}(\Delta, \mathbb{R})$ *is a-closed.*

(b) Δ *contains no copy of* Λ_2.

(c) *The lateral completion of* $F(\Delta, \mathbb{R})$ *is* $V(\Delta, \mathbb{R})$.

Proof. (a) $\Rightarrow$ (b) Suppose that Δ does contain a copy of Λ_2. Let x be the characteristic function of this copy. Then $x \in D_{\omega_1}(\Delta, \mathbb{R}) \setminus C_{\omega_1}(\Delta, \mathbb{R})$, contradicting the fact that $C_{\omega_1}(\Delta, \mathbb{R})$ is a-closed.

(b) $\Rightarrow$ (a) Suppose that $C_{\omega_1}(\Delta, \mathbb{R})$ has a proper abelian a-extension $H \subseteq V(\Delta, \mathbb{R})$. Let $0 < h \in H \setminus C_{\omega_1}(\Delta, \mathbb{R})$; then there exist uncountably many maximal chains in $\mathrm{supp}(h)$.

In $\mathrm{supp}(h)$, define $a \sim \beta$ if α and β lie on the same maximal chains of $\mathrm{supp}(h)$. This is an equivalence relation; for each equivalence class $[\alpha]$, let $\overline{\alpha}$ be the maximal element in $[\alpha]$; $\overline{\mathrm{supp}}(h)$ will denote the set $\{\overline{\alpha}\colon \alpha \in \mathrm{supp}(h)\}$. $\overline{\mathrm{supp}}(h)$ is easily shown to satisfy the ascending chain condition, and if $\alpha < \beta$ in $\mathrm{supp}(h)$, then $\overline{\alpha} \leq \overline{\beta}$ in $\overline{\mathrm{supp}}(h)$. If $\{\delta_\lambda\}$ is an antichain in $\mathrm{supp}(h)$, then $\{\overline{\delta_\lambda}\}$ is an antichain in $\overline{\mathrm{supp}}(h)$. Moreover, for any $\delta \in \mathrm{supp}(h)$, the cardinality of maximal chains of $\mathrm{supp}(h)$ containing δ is equal to the cardinality of maximal chains of $\overline{\mathrm{supp}}(h)$ containing $\overline{\delta}$. Most important, if $\overline{\alpha} < \overline{\beta}$, then there exists $\overline{\gamma} \in \overline{\mathrm{supp}}(h)$ such that $\overline{\gamma} < \overline{\beta}$ and $\overline{\gamma}$ is incomparable to $\overline{\alpha}$.

Let $U = \{\overline{\delta} \in \overline{\mathrm{supp}}(h) : \overline{\delta}$ lies on uncountably many maximal chains of $\overline{\mathrm{supp}}(h)\}$. We claim that for any $\overline{\delta} \in U$, there exist $\overline{\alpha}, \overline{\beta} \in U$ such that $\overline{\alpha}$ is incomparable to $\overline{\beta}$ and $\overline{\alpha}, \overline{\beta} < \overline{\delta}$.

Suppose not. Let $\overline{\delta_0} = \overline{\delta}$. Since $\overline{\delta_0} \in U$, there exists $\overline{\alpha} \in \overline{\mathrm{supp}}(h)$ such that $\overline{\alpha} < \overline{\delta_0}$. Let children$(\overline{\delta_0})$ denote the set of elements of $\overline{\mathrm{supp}}(h)$ maximal below $\overline{\delta_0}$. By our assumption, there exists a unique $\alpha \in$ children$(\overline{\delta_0}) \cap U$; let $\overline{\delta_1}$ be this $\overline{\alpha}$.

Now let τ be a countable ordinal such that for any ordinal $\sigma < \tau$, $\overline{\delta_\sigma}$ has been defined such that $\overline{\delta_\sigma} \in U$, and $\{\overline{\delta_\sigma}\}_{\sigma < \tau}$ is a chain where if $\sigma_1 < \sigma_2$, then $\overline{\delta_{\sigma_1}} > \overline{\delta_{\sigma_2}}$.

If $\tau = \sigma + 1$ is a successor ordinal, then let $\overline{\delta_\tau}$ be the unique element of children$(\overline{\delta_\sigma} \cap U)$. If τ is a limit ordinal, then for any $\sigma < \tau$, let $\overline{\Delta_\sigma} = \{\gamma \in \operatorname{supp}(h) : \gamma \leq \overline{\delta_\sigma}\}$. Let $\Sigma_\tau = \bigcap_{\sigma < \tau} \overline{\Delta_\sigma}$. Then $h - h|_{\Sigma_\tau} \in C_{\omega_1}(\Delta, \mathbb{R})$, implying that $h|_{\Sigma_\tau} \notin C_{\omega_1}(\Delta, \mathbb{R})$. But then by our assumption, there exists a unique maximal component of $\overline{\operatorname{supp}}(h|_{\Sigma_\tau})$ that is also in U; let $\overline{\delta_\tau}$ be this element.

We can clearly continue in this way until, for any ordinal $\tau < \omega_1$, we have constructed a chain $\{\overline{\delta_\tau}\}$. Now for each $\tau < \omega_1$, there exists a maximal $\overline{\alpha_\tau} \in \overline{\operatorname{supp}}(h)$ such that $\overline{\alpha_\tau} < \overline{\delta_\tau}$ and $\overline{\alpha_\tau}$ is incomparable to $\overline{\delta_{\tau+1}}$. But $\{\overline{\alpha_\tau}\}$ forms an uncountable antichain and are maximal elements of $h - h|_{([\delta_0], \dots, [\delta_\tau], \dots)} \in C_{\omega_1}(\Delta, \mathbb{R})$. This contradicts the fact that H is an a-extension of $C_{\omega_1}(\Delta, \mathbb{R})$, which is countably Δ-valued.

Thus the claim above is true. Choose $\overline{\delta} \in \overline{\operatorname{supp}}(h) \cap U$, and choose incomparable $\overline{\alpha}, \overline{\beta} \in \overline{\operatorname{supp}}(h) \cap U$ below $\overline{\delta}$. Below each of these, choose similar elements. By continuing in this way, we obtain a copy of Λ_2.

(b) $\Rightarrow$ (c) Let $0 < v$ be a special element in $V(\Delta, \mathbb{R})$ and $\delta \in \operatorname{supp}(v)$; let $[\delta]$ be the principal order ideal of $\operatorname{supp}(v)$ generated by $\{\delta\}$.

Now if below each $\delta \in \operatorname{supp}(v)$, there are two incomparable support elements, then $\operatorname{supp}(v)$ contains a copy of Λ_2. So there exists $\delta \in \operatorname{supp}(v)$ such that $[\delta]$ is totally ordered and hence $v|_{[\delta]} \in F(\Delta, \mathbb{R})$.

Let $A = \{h \in F(\Delta, \mathbb{R}))^L : \operatorname{supp}(h)$ is an order ideal of $\operatorname{supp}(v)$ and $h(\delta) = v(\delta)$ for all $\delta \in \operatorname{supp}(h)\}$. By the paragraph above, $A \neq \emptyset$. Let $\Omega = \bigcup_{h \in A} \operatorname{supp}(h)$. Define $y : \Delta \to \mathbb{R}$ by

$$y(\delta) = \begin{cases} v(\delta), & \text{if } \delta \in \Omega, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\operatorname{supp}(y) = \Omega$ and so $y \in V(\Delta, \mathbb{R})$. Also $\operatorname{supp}(y)$ is an order ideal of $\operatorname{supp}(v)$, and y agrees with v on its support. So to show that $y \in A$, we must show that $y \in F(\Delta, \mathbb{R})^L$.

Let δ be a maximal element in $\operatorname{supp}(y) = \Omega$. Then δ must be a maximal component for some $h \in A$, and so $y|_{[\delta]} = h|_{[\delta]} \in F(\Delta, \mathbb{R})^L$. Since all special components of y are in $F(\Delta, \mathbb{R})^L$, $y \in F(\Delta, \mathbb{R})^L$. Hence, $y = \bigvee A$.

Suppose that $v \neq y$. Since $v > y$, let $\delta \in \operatorname{supp}(v - y)$ such that $z = (v - y)|_{[\delta]} \in F(\Delta, \mathbb{R})$. As $0 < z \leq v - y$, $y < z + y \leq v$. But clearly $z + y \in A$, which contradicts the fact that $y = \bigvee A$. So $v = y$.

Thus all special elements of $V(\Delta, \mathbb{R})$ are in $F(\Delta, \mathbb{R})^L$; since $V(\Delta, \mathbb{R})$ is laterally complete, we must have equality.

(c) $\Rightarrow$ (b) Let $V_{\Lambda_2} = \{v \in V(\Delta, \mathbb{R}) : \operatorname{supp}(v)$ contains no copy of $\Lambda_2\}$. Now suppose there exist $v, w \in V_{\Lambda_2}$ such that there exists a copy Ω of Λ_2 in $\operatorname{supp}(v + w)$.

For $\delta \in \Delta$, let $[\delta]$ be the principal order ideal of Δ generated by δ. Since $v \in V_{\Lambda_2}$, there exists $\delta \in \Omega$ such that $[\delta] \cap \operatorname{supp}(v)$ is either empty or totally ordered. If empty, then $\Omega \cap [\delta] \subseteq \operatorname{supp}(w)$, and so $\operatorname{supp}(w)$ contains a copy of Λ_2, contradicting $w \in V_{\Lambda_2}$. If $[\delta] \cap \operatorname{supp}(v)$ is totally ordered, there there exists $\beta \in \Omega$ incomparable to $[\delta] \cap \operatorname{supp}(v)$, and so $\Omega \cap [\beta] \subseteq \operatorname{supp}(w)$ again contains a copy of Λ_2. Thus $v + w \in V_{\Lambda_2}$, and clearly for any $v \in V_{\Lambda_2}$, $v \vee 0 \in V_{\Lambda_2}$. So V_{Λ_2} is an l-subgroup of $V(\Delta, \mathbb{R})$. Clearly V_{Λ_2} is laterally complete and $F(\Delta, \mathbb{R})$ is dense in V_{Λ_2}. Thus $V(\Delta, \mathbb{R}) = F(\Delta, \mathbb{R})^L \subseteq V_{\Lambda_2} \subseteq V(\Delta, \mathbb{R})$ shows that Δ contains no copy of Λ_2. ∎

A few remarks about $C_{\omega_1}(\Lambda_2, \mathbb{R})$ and $D_{\omega_1}(\Lambda_2, \mathbb{R})$ are in order. First, it is easy to see that $D_{\omega_1}(\Lambda_2, \mathbb{R}) = V(\Lambda_2, \mathbb{R})$, and so in this case, $V(\Lambda_2, \mathbb{R})$ is the a-closure of $C_{\omega_1}(\Lambda_2, \mathbb{R})$. Second, since any antichain in Λ_2 is countable, $C_{\omega_1}(\Lambda_2, \mathbb{R})$ is laterally complete and so contains the lateral completion of $F(\Lambda_2, \mathbb{R})$. As yet, there is no explicit description of $F(\Lambda_2, \mathbb{R})$.

It is possible to generalize on condition (c) of Theorem 4.27. Let Δ and Λ be root systems. Define $V_\Lambda(\Delta, \mathbb{R}) = \{v \in V(\Delta, \mathbb{R}) : \operatorname{supp}(v)$ contains no copy of $\Lambda\}$. For $V_\Lambda(\Delta, \mathbb{R})$ to be a proper l-subgroup of $V(\Delta, \mathbb{R})$, we first must have that Λ satisfies the ascending chain condition, as otherwise since for every $v \in V(\Delta, \mathbb{R})$, that $\operatorname{supp}(v)$ satisfies the ascending chain condition would place all of $V(\Delta, \mathbb{R})$ into $V_\Lambda(\Delta, \mathbb{R})$. Also, in general, for $V_\Lambda(\Delta, \mathbb{R})$ to be an l-subgroup of $V(\Delta, \mathbb{R})$ requires the following.

PROPOSITION 4.28. *Let Λ be a root system satisfying the ascending chain condition. Then for any root system Δ, $V_\Lambda(\Delta, \mathbb{R})$ is an l-subgroup of $V(\Delta, \mathbb{R})$ if and only if $\Lambda = \Lambda_A \cup \Lambda_B$ implies that Λ_A or Λ_B contains a copy of Λ.*

Proof. For one direction, suppose that $\Lambda = \Lambda_A \cup \Lambda_V$, where neither Λ_A nor Λ_B contains a copy of Λ. Then in $V(\Lambda, \mathbb{R})$, let a be the characteristic function of Λ_A and b be the characteristic function of Λ_B. Then $a, b \in V_\Lambda(\Lambda, \mathbb{R})$ but $a + b \notin V_\Lambda(\Lambda, \mathbb{R})$.

Conversely, it is easy to see that if $v \in V_\Lambda(\Delta, \mathbb{R})$, then $v \vee 0 \in V_\Lambda(\Delta, \mathbb{R})$. Now suppose there exist $g, h \in V_\Lambda(\Delta, \mathbb{R})$ such that $g + h \notin V_\Lambda(\Delta, \mathbb{R})$. Then $\operatorname{supp}(g + h)$ contains a copy Λ' of Λ. Let $\Lambda_A = \Lambda' \cap \operatorname{supp}(g)$ and $\Lambda_B = \Lambda' \cap \operatorname{supp}(h)$. Then $\Lambda' \subseteq \Lambda_A \cup \Lambda_B$ implies that either Λ_A or Λ_B contains a copy of Λ, contradicting the fact that neither g nor h is in $V_\Lambda(\Delta, \mathbb{R})$. So $g + h \in V_\Lambda(\Delta, \mathbb{R})$. ∎

DEFINITION 11. A root system Λ is *secure* if Λ satisfies the ascending chain condition and if $\Lambda = \Lambda_A \cup \Lambda_B$, then either Λ_A or Λ_B contains a copy of Λ.

PROPOSITION 4.29. *For any secure root system Λ that is not trivially ordered, $V(\Delta, \mathbb{R})$ is the abelian Δ-closure of $V_\Lambda(\Delta, \mathbb{R})$.*

Proof. For any antichain $\Omega \subseteq \Delta$, the characteristic function $c_\Omega \in V_\Lambda(\Delta, \mathbb{R})$. Consequently, $\Delta_{V_\Lambda} = T_\Delta$, and so we are done. ∎

5. UNSOLVED QUESTIONS

Below we list some questions about a-closed and l-groups and Δ-closed l-groups.

(1) If C is a convex l-subgroup of an a-closed l-group G, is C a-closed?

(2) What are the Δ-closures of $\mathbb{R}^{\mathrm{EC}}$?

(3) If T is an a-closed o-group, is $V(\Delta, T)$ a-closed?

(4) If G has a unique a-closure H, is H^{EC} the unique a-closure of G^{EC}?

(5) For any l-group G and plenary subset Δ of $\Gamma(G)$, does G have a Δ-closure?

(6) If A is an abelian l-group that is Δ-closed in the category of abelian l-groups with respect to some plenary subset Δ of $\Gamma(A)$, is A Δ-closed in the category of all l-groups? (We have shown that A is Δ-closed in the category of normal-valued l-groups.)

(7) For any topological space X, is $C(X)$ X-closed?

(8) If G is a finite-valued vector lattice, then G can be v-embedded into $V(\gamma(G, \mathbb{R})$ as a sublattice. Is each a-closure of G a vector lattice and, if so, is each a-closure a sublattice of $V(\Gamma(G), \mathbb{R})$?

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